Name: ____

Exam 3

Math 2130 — Fall 2022

4 November, 2022

Instructions: Work alone, using only paper and a writing implement. Make sure your name is on every page. Remember that your grade will be based on the understanding you demonstrate, not just the correctness of your answer, so be sure to include comprehensible justification for your work.

1. Find the area of the triangle in the plane with vertices $P = (2,3)$, $Q = (-1,-1)$, and $R = (4,-2)$.

   Solution. It is half the area of the parallelogram spanned by the difference vectors $Q - P = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$ and $R - P = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$. We compute this with a determinant:

   $$\det \begin{pmatrix} -3 & 2 \\ -4 & -5 \end{pmatrix} = 15 + 8 = 23$$

   Therefore the area of the triangle is $\frac{23}{2}$. □

2. Compute the determinant of the following matrix, by any method.

   $$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 8 & 1 & 2 & 3 \\ 1 & 4 & 2 & 3 & 0 \\ 1 & 2 & 1 & 2 & 1 \end{pmatrix}$$

   Solution. This matrix is block lower triangular, so its determinant coincides with

   $$\begin{aligned} \det \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \det \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix} &= -\det \begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \\ &= -\det \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\ &= -\det \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\ &= -\det \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ &= -\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \\ &= -2 \end{aligned}$$

   □

3. Compute the contribution of the permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

to the 'big formula' for the determinant of the matrix A from the last problem.

Solution. The contribution is $1(1)(3)(2)(1)\det(P) = 6\det(P) = -6$ (since it takes 3 row swaps to convert P to I). $\square$

Instructions: Do at least two of the following six problems. Write your answers on separate paper. Do not forget to write your name on the additional pages you submit.

4. In this problem $A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix}$ is an unknown 3×3 matrix with determinant 4. Let $\vec{x}$ be the following vector

$$\vec{x} = \begin{pmatrix} 5 \\ 0 \\ 7 \end{pmatrix}$$

and let $\vec{b} = A\vec{x}$. Compute the determinant of the matrix $\begin{pmatrix} 2\vec{v}_2 + 3\vec{v}_3 & \vec{b} & \vec{v}_1 \end{pmatrix}$.

Solution. We can write $\vec{b} = 5\vec{v}_1 + 7\vec{v}_3$. Then we can compute

$$\begin{aligned} \det\begin{pmatrix} 2\vec{v}_2 + 3\vec{v}_3 & 5\vec{v}_1 + 7\vec{v}_3 & \vec{v}_1 \end{pmatrix} &= \det\begin{pmatrix} 2\vec{v}_2 + 3\vec{v}_3 & 7\vec{v}_3 & \vec{v}_1 \end{pmatrix} \\ &= \det\begin{pmatrix} 2\vec{v}_2 & 7\vec{v}_3 & \vec{v}_1 \end{pmatrix} \\ &= 14\det\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} \\ &= 56 \end{aligned}$$

$\square$

5. The following matrix has orthogonal (not orthonormal) columns and *positive* determinant, but some of its entries are unknown. The unknown entries are labelled with an asterisk ($*$).

$$A = \begin{pmatrix} * & 2 & 4 & * \\ * & * & * & * \\ 1 & 0 & * & -2 \\ * & 3 & * & * \end{pmatrix}$$

The lengths of the columns of A are (in order)

$$3, 10, 6, 5.$$

Write down as many entries of its cofactor matrix as you can.

Solution. Since A is orthogonal

$$A^T A = \begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 36 & 0 \\ 0 & 0 & 0 & 25 \end{pmatrix}.$$

Therefore $\det(A)^2 = 9(100)(36)(25)$ so $\det(A) = 3(10)(6)(5) = 900$ (we can rule out -900 because $\det(A) > 0$).

If C is the cofactor matrix of A, then

$$CA^T = \det(A)I = \begin{pmatrix} 900 & 0 & 0 & 0 \\ 0 & 900 & 0 & 0 \\ 0 & 0 & 900 & 0 \\ 0 & 0 & 0 & 900 \end{pmatrix}.$$

Multiplying both sides by A on the right, we get

$$CA^T A = 900A.$$

Now we can get C:

$$C = 900A(A^T A)^{-1} = 900A \begin{pmatrix} \frac{1}{9} & 0 & 0 & 0 \\ 0 & \frac{1}{100} & 0 & 0 \\ 0 & 0 & \frac{1}{36} & 0 \\ 0 & 0 & 0 & \frac{1}{25} \end{pmatrix} = \begin{pmatrix} * & 2 & 4 & * \\ * & * & * & * \\ 1 & 0 & * & -2 \\ * & 3 & * & * \end{pmatrix} \begin{pmatrix} 100 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix}$$
$$= \begin{pmatrix} * & 18 & 100 & * \\ * & * & * & * \\ 100 & 0 & * & -72 \\ * & 27 & * & * \end{pmatrix}$$

□

6. Suppose that A is a square matrix whose entries are all *integers*. For what values of $\det(A)$ does A have an invese whose entries are all integers? Explain your answer.

   Solution. If A^{-1} consists only of integers then $\det(A)$ and $\det(A^{-1})$ will be two integers that multiply to 1. Therefore they must both be 1 or both be -1.

   On the other hand, if A consists only of integers then its cofactor matrix C also consists only of integers. Therefore, if $\det(A)$ is 1 or -1 then $A^{-1} = \det(A)^{-1}C^T$ consists only of integers.

   Thus A^{-1} is a matrix of integers if and only if $\det(A)$ is 1 or -1. □

7. The following matrix has n rows and n columns, following the pattern begun below. Find a pattern relating $\det(A_n)$ to $\det(A_{n-1})$ and $\det(A_{n-2})$. Use it to compute $\det(A_{100})$.

$$A_n = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \cdots \\ -1 & 2 & -1 & 0 & 0 & \cdots \\ 0 & -1 & 2 & -1 & 0 & \cdots \\ 0 & 0 & -1 & 2 & -1 & \cdots \\ 0 & 0 & 0 & -1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

   Solution. If we expand $\det(A_n)$ along the first row, we get

$$\det(A_n) = 2\det(A_{n-1}) - (-1)\det \begin{pmatrix} -1 & -1 & 0 & 0 & \cdots \\ 0 & 2 & -1 & 0 & \cdots \\ 0 & -1 & 2 & -1 & \cdots \\ 0 & 0 & -1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

   The last matrix is block upper triangular, so we get

$$\det(A_n) = 2\det(A_{n-1}) - \det(A_{n-2}).$$

Now we can compute $\det(A_{100})$:

$$\begin{aligned}
\det(A_1) &= 2\\
\det(A_2) &= 3\\
\det(A_3) &= 2(3) - 2 = 4\\
\det(A_4) &= 2(4) - 3 = 5
\end{aligned}$$

Thus $\det(A_n) = n + 1$ for every n. (To see that this is true, observe that $f(n) = n + 1$ satisfies the relation $f(n) = 2f(n-1) - f(n-2)$: we have $n + 1 = 2n - (n-1)$.) So $\det(A_{100}) = 101$. □

8. For what values of c is the following matrix invertible?

$$\begin{pmatrix} -c & 2 & 3 \\ 0 & -6-c & -1 \\ 0 & 6 & -1-c \end{pmatrix}$$

Solution. We calculate the determinant. It is

$$-c\big((-6-c)(-1-c) - 3(6)\big) = -c(c^2 + 7c + 12) = -c(c+3)(c+4).$$

This will be zero if c is 0, -3, or -4. Therefore the matrix is invertible if c is any number other than 0, -3, or -4. □

9. In this problem, A will be the 3×4 matrix shown below and B will be an unknown 4×3 matrix.

$$A = \begin{pmatrix} 2 & 3 & 0 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{pmatrix}$$

What we know about B are the determinants of its 3×3 minors:

$$B_{123} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ \cancel{b_{41}} & \cancel{b_{42}} & \cancel{b_{43}} \end{pmatrix} \qquad \det(B_{123}) = -7$$

$$B_{124} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ \cancel{b_{31}} & \cancel{b_{32}} & \cancel{b_{33}} \\ b_{41} & b_{42} & b_{43} \end{pmatrix} \qquad \det(B_{124}) = -1$$

$$B_{134} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ \cancel{b_{21}} & \cancel{b_{22}} & \cancel{b_{23}} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{pmatrix} \qquad \det(B_{134}) = 1$$

$$B_{234} = \begin{pmatrix} \cancel{b_{11}} & \cancel{b_{12}} & \cancel{b_{13}} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{pmatrix} \qquad \det(B_{234}) = 0$$

Calculate the determinant of AB.

Solution. The determinant of AB will not be affected by row operations on A, so it is the same as $\det(A'B)$ where

$$A' = \begin{pmatrix} 2 & 3 & 0 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The determinants of the 3×3 minors of A' are

$$\begin{aligned}
\det(A_{123}) &= 0 \\
\det(A_{124}) &= 2 \\
\det(A_{134}) &= 6 \\
\det(A_{234}) &= 9
\end{aligned}$$

The determinant of AB is therefore

$$\begin{aligned}
&\det(A_{123})\det(B_{123}) + \det(A_{124})\det(B_{124}) + \det(A_{134})\det(B_{134}) + \det(A_{234})\det(B_{234}) \\
&\qquad = 0(-7) + 2(-1) + 6(1) + 9(0) = 4
\end{aligned}$$

$\square$