Name: ______________________________

Exam 2

Math 2130 — Fall 2022

17 October, 2022

Instructions: Work alone, using only paper and a writing implement. Make sure your name is on every page. Some of the problems below may have more than one correct solution, and some solutions may be more efficient than others, so spend some time considering the possible approaches before starting to calculate. Remember that your grade will be based on the understanding you demonstrate, not just the correctness of your answer, so be sure to include comprehensible justification for your work.

1. Let V be the hyperplane in $\mathbb{R}^5$ with the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 0$.

   (a) Write the vector

   $$\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$

   as the sum of a vector in V and a vector in $V^\perp$. (Hint: we have discussed many ways to do this. Some approaches may be quicker than others.)

   Solution. The hyperplane V is orthogonal to the line spanned by $\vec{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. The projection of $\vec{b}$ on $V^\perp$ is $\vec{z} = \frac{\vec{w}^T\vec{b}}{\vec{w}^T\vec{w}}\vec{w} = \frac{15}{5}\vec{w} = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}$. Thus the projection of $\vec{b}$ on V is $\vec{b} - \vec{z} = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$ and

   $$\vec{b} = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}$$

   □

   (b) Describe some ways you can check your answer.
   Solution. We can check that $\vec{y}^T\vec{z} = 0$, that $\vec{z}$ is a multiple of $(1, 1, 1, 1, 1)$, that $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix} \vec{y} = 0$. □

2. Find an equation whose solutions are the coefficients a and b in the equation $y = ax + b$ of the line that best fits the points $P_1 = (0, 0)$, $P_2 = (1, 1)$, $P_3 = (2, 4)$, and $P_4 = (3, 9)$. The meaning of 'best' here is that a and b should minimize the length of the following vector:

   $$\begin{pmatrix} ax(P_1) + b - y(P_1) \\ ax(P_2) + b - y(P_2) \\ ax(P_3) + b - y(P_3) \\ ax(P_4) + b - y(P_4) \end{pmatrix}$$

Your answer should consist of a matrix C and a vector $\vec{d}$ such that $\begin{pmatrix} a \\ b \end{pmatrix}$ is the solution to the equation $C\begin{pmatrix} a \\ b \end{pmatrix} = \vec{d}$. You do not have to solve this equation.

Solution. We are trying to solve the equation $A\begin{pmatrix} a \\ b \end{pmatrix} = \vec{b}$ where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \qquad \text{and} \qquad \vec{b} = \begin{pmatrix} 0 \\ 1 \\ 4 \\ 9 \end{pmatrix}$$

The nearest approximate solution is the solution to $A^T A\begin{pmatrix} a \\ b \end{pmatrix} = A^T\vec{b}$. Therefore

$$C = A^T A = \begin{pmatrix} 14 & 6 \\ 6 & 4 \end{pmatrix} \qquad \text{and} \qquad \vec{b} = \begin{pmatrix} 36 \\ 14 \end{pmatrix}$$

□

3. Suppose that V and W are subspaces of $\mathbb{R}^n$ and $V \cap W$ is *not* the zero subspace. Let P be the projection matrix onto V and let Q be the projection matrix onto W. Mark each of the following statements as *always*, *sometimes*, or *never* true. Explain your answers.

   (a) $P + Q$ is the projection matrix for $V + W$. (Recall that $V + W$ consists of all vectors that can be written as a sum of a vector in V and a vector in W.)

   Solution. This is never the case. If $\vec{v}$ is in $V \cap W$ then $(P + Q)\vec{v} = 2\vec{v}$ but the projection of $\vec{v}$ should be $\vec{v}$ because $\vec{v}$ is in $V \cap W$. □

   (b) PQ is the projection matrix for $V \cap W$. (Recall that $V \cap W$ is the subspace consisting of all vectors that are in both V and W.)

   Solution. This is sometimes true. It will be true if V and W are perpendicular to each other (meaning every vector in W is the sum of a vector in $V \cap W$ and a vector in $V^\perp$). Otherwise it will be false because in that case, if $\vec{w}$ is in W but not in V then $PQ\vec{w} = P\vec{w}$ will be in V but not in W. (If it were also in W then the vector $\vec{w} - P\vec{w}$ would be orthogonal to V and in W, so V would be perpendicular to W.)

   Another explanation: A matrix R is an orthogonal projection if and only if $R^T = R$ and $R^2 = R$. We know that $P^2 = P^T = P$ and $Q^2 = Q^T = Q$. If $PQ = QP$ then $(PQ)^2 = PQPQ = P^2Q^2 = PQ$ and $(PQ)^T = Q^T P^T = QP = PQ$. If $(PQ)^T = PQ$ then $QP = Q^T P^T = (PQ)^T = PQ$. In this case, we can show V and W are perpendicular: if $\vec{v} \in V$ and $\vec{w} \in W$ then $Q\vec{v} = QP\vec{v}$ and $P\vec{w} = PQ\vec{w}$ are in $V \cap W$ so $\vec{v} - Q\vec{v}$ is in V and $\vec{w} - P\vec{w}$ is in W; these are orthogonal because $(\vec{v} - Q\vec{v})^T(\vec{w} - P\vec{w}) = \vec{v}^T\vec{w} - \vec{v}^T P\vec{w} - \vec{v}^T Q\vec{w} + \vec{v}^T QP\vec{w} = \vec{v}^T\vec{w} - \vec{v}^T\vec{w} - \vec{v}^T\vec{w} + \vec{v}^T\vec{w} = \vec{0}$. □

   (c) $I - P$ is the projection matrix for $V^\perp$. (Recall that $V^\perp$ consists of all vectors that are orthogonal to V.)

   Solution. This is always true. For any vector $\vec{b}$, the projection of $\vec{b}$ on $V^\perp$ is $\vec{b} - P\vec{b} = (I - P)\vec{b}$. □

4. The matrices below satisfy the equation $A = BC$ and the columns of B are orthogonal (not orthonormal). Calculate C. (Hint: can you find a left inverse of B?)

$$A = \begin{pmatrix} 11 & -8 \\ -9 & 7 \\ 5 & -4 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 1 \\ -2 & 1 \\ 1 & -1 \end{pmatrix}$$

Solution. Since B has orthogonal columns, $B^T B$ is a diagonal matrix whose diagonal entries are the squares of the lengths of the columns of B:

$$B^T B = \begin{pmatrix} 14 & 0 \\ 0 & 3 \end{pmatrix}$$

Therefore $B^T A = B^T B C$, so

$$
\begin{aligned}
C = (B^T B)^{-1} B^T A &= \begin{pmatrix} \frac{1}{14} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 3 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 11 & -8 \\ -9 & 7 \\ 5 & -4 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{14} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 56 & -42 \\ -3 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}
\end{aligned}
$$

□