Name:

Exam 1

Math 2130 — Fall 2022

16 September, 2022

Instructions: Work alone, using only paper and a writing implement. Make sure your name is on every page. There are many ways to solve the problems below. Some involve much less calculation than others, so spend some time considering the possible approaches before starting to calculate. Remember that your grade will be based on the understanding you demonstrate, not just the correctness of your answer, so be sure to include comprehensible justification for your work.

1. The following two matrices, A and B, include some known and some unknown entries. Unknown entries are marked with an asterisk ($*$). Compute all entries of AB that can be determined from the given information. Place an asterisk ($*$) in the entries that cannot be determined. Correct answers do not require additional explanation.

$$A = \begin{pmatrix} 2 & 1 \\ * & 0 \\ -5 & * \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -1 & 2 \\ * & 3 & 0 \end{pmatrix}$$

Solution.

$$AB = \begin{pmatrix} * & 1 & 4 \\ 0 & * & * \\ * & * & -10 \end{pmatrix}$$

□

2. The matrix on the left has two unknown entries, a and b. The matrix on the right is its reduced row echelon form. Determine a and b. Show how you arrive at your answer.

$$\operatorname{rref}\begin{pmatrix} 3 & -2 & a \\ 3 & -4 & -4 \\ b & 2 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{pmatrix}$$

Solution. Since the linear relationships between the columns of the reduced row echelon form are the same as the linear relationships between the columns of the original matrix, we must have

$$8\begin{pmatrix} 3 \\ 3 \\ b \end{pmatrix} + 7\begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ -4 \\ 6 \end{pmatrix}.$$

Therefore,

$a =$ 10 $\qquad\qquad$ $b =$ -1

□

3. For what values of a is the following matrix invertible? Show how you arrive at your answer.

$$\begin{pmatrix} 2 & 1 & 0 \\ -4 & -1 & 1 \\ 2 & 6 & a \end{pmatrix}$$

Solution. We row reduce the matrix. After adding twice the first row to the second row and subtracting the first from from the third row, we get

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 5 & a \end{pmatrix}.$$

Now subtract five times the second row from the third row to get

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & a-5 \end{pmatrix}$$

This matrix is in row echelon form. It has 3 pivots if and only if a is not 5. Therefore the matrix is invertible if a is not 5. □

4. Find the matrix A such that $AB = C$, then find A^2B. Hint: think about row operations. Show how you arrive at your answer.

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 8 & 16 \\ 3 & 9 & 27 & 81 \\ 4 & 16 & 64 & 256 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 7 & 15 \\ 1 & 5 & 19 & 65 \\ 1 & 7 & 37 & 175 \end{pmatrix}$$

Solution. The first row of C is the same as the first row of B. The second row of C is the difference of the second row of B and the first row; the third row of C is the difference of the third and second rows of B; the fourth row of C is the difference of the third and fourth rows of B. The matrix that performs these row operations is

$$A = \begin{pmatrix} \boxed{1} & \boxed{0} & \boxed{0} & \boxed{0} \\ \boxed{-1} & \boxed{1} & \boxed{0} & \boxed{0} \\ \boxed{0} & \boxed{-1} & \boxed{1} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{-1} & \boxed{1} \end{pmatrix}$$

We can get $A^2B = AC$ by performing the same row operations on C. We get

$$A^2B = AC = \begin{pmatrix} \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{0} & \boxed{2} & \boxed{6} & \boxed{14} \\ \boxed{0} & \boxed{2} & \boxed{12} & \boxed{50} \\ \boxed{0} & \boxed{2} & \boxed{18} & \boxed{110} \end{pmatrix}$$

□