Theorem 1. *Every natural number is even or odd.*

Proof. We use the method of smallest counterexample to show that it is impossible for a natural number to be neither even nor odd. Suppose that there is a natural number that is neither even nor odd. Then there must be a smallest natural number that is neither even nor odd. Suppose that y is this natural number.

It is not possible for y to be 1, because we know $1 = 2 \times 0 + 1$, so by the definition of oddness, 1 is an odd number. Since $y \neq 1$, we know $y > 1$ so $y - 1 > 0$; thus $y - 1$ is a natural number. We assumed that y was the smallest natural number that was neither even nor odd. Since $y - 1$ is a natural number, this means $y - 1$ must be even or odd. We consider these possibilities separately.

If $y - 1$ is even then $y = (y - 1) + 1$ is the sum of an even integer and an odd integer, so y is odd. This contradicts the assumption that y was neither even nor odd.

On the other hand, if $y - 1$ is odd then $y = (y - 1) + 1$ is the sum of two odd integers, so y is even. This also contradicts the assumption that y was neither even nor odd.

Either way, we obtained an unreasonable conclusion. Thus our original assumption, that there was a natural number that was neither even nor odd, must have been impossible. □

Proof. We prove that every natural number n is even or odd by strong induction on n.

BASE CASE: $n = 1$. We know that $1 = 2 \times 0 + 1$ so 1 is odd, by definition of oddness.

INDUCTION STEP: $n > 1$. We assume for the sake of induction that we already know every natural number $m < n$ is either even or odd and we use this to prove that n is even or odd. Since $n > 1$, we know $n - 1 > 0$, so $n - 1$ is a natural number. Therefore, by the inductive hypothesis, $n - 1$ is either even or odd. If $n - 1$ is even then we observe that $n = (n - 1) + 1$, so n is the sum of an even number and an odd number, hence is odd; if $n - 1$ is odd then $n = (n - 1) + 1$ is the sum of two odd numbers, hence is even. Either way, n is an even number or an odd number, which is what we had to show.

By strong induction, we may now conclude that every natural number n is even or odd. □

Proof. We prove that every natural number n is even or odd by induction on n.

BASE CASE: $n = 1$. We know that $1 = 2 \times 0 + 1$ so 1 is odd, by definition of oddness.

INDUCTION STEP: $n > 1$. We assume for the sake of induction that we already know $n - 1$ is either even or odd. If $n - 1$ is even then we observe that $n = (n - 1) + 1$, so n is the sum of an even number and an odd number, hence is odd; if $n - 1$ is odd then $n = (n - 1) + 1$ is the sum of two odd numbers, hence is even. Either way, n is an even number or an odd number, which is what we had to show.

By induction, we may now conclude that every natural number n is even or odd. □