Theorem 1. *Let ℓ, m, and n be integers such that the following inequalities hold:*

I1: $m \geq 0$

I2: $n \geq 0$

I3: $\ell \geq m$

I4: $\ell \geq n$

I5: $\ell \leq m + n$

Then there are sets A and B such that $|A| = n$ and $|B| = m$ and $|A \cup B| = \ell$.

Proof:

We choose the following sets for A and B:

$$A = \{1, 2, \ldots, n\}$$
$$B = \{n+1, n+2, \ldots, \ell, 1, 2, \ldots, m+n-\ell\}$$

To complete the proof, we have to check that A, B, and $A \cup B$ have the right sizes. We can see that

$$|A| = n$$

since A contains the first n positive integers.

We can also see that

$$|B| = (\ell - n) + (m + n - \ell) = m$$

since B contains the $\ell - n$ integers between $n + 1$ and ℓ, inclusive, and the first $m + n - \ell$ integers.

To compute $|A \cup B|$, first we write out the elements of $A \cup B$:

$$A \cup B = \{1, 2, \ldots, n, n+1, n+2, \ldots, \ell, 1, 2, \ldots, m+n-\ell\}$$

The elements $1, 2, \ldots, m + n - \ell$ are listed twice,

so we can remove the second appearance:

$$A \cup B = \{1, 2, \ldots, \ell\}.$$

Thus $A \cup B$ consists of the first ℓ positive integers, so it has size ℓ. This completes the proof. $\square$