f. 1,000,000.
g. 10^n where n is a positive integer.
h. $30 = 2 \times 3 \times 5$.
i. $42 = 2 \times 3 \times 7$. (Why do 30 and 42 have the same number of positive divisors?)
j. $2310 = 2 \times 3 \times 5 \times 7 \times 11$.
k. $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8$.
l. 0.

3.13. An integer n is called *perfect* provided it equals the sum of all its divisors that are both positive and less than n. For example, 28 is perfect because the positive divisors of 28 are 1, 2, 4, 7, 14, and 28. Note that $1 + 2 + 4 + 7 + 14 = 28$.
a. There is a perfect number smaller than 28. Find it.
b. Write a computer program to find the next perfect number after 28.

3.14. *At a Little League game there are three umpires. One is an engineer, one is a physicist, and one is a mathematician. There is a close play at home plate, but all three umpires agree the runner is out.*

Furious, the father of the runner screams at the umpires, "Why did you call her out?!"

The engineer replies, "She's out because I call them as they are."
The physicist replies, "She's out because I call them as I see them."
The mathematician replies, "She's out because I called her out."
Explain the mathematician's point of view.

4 Theorem

A *theorem* is a declarative statement about mathematics for which there is a proof.

The notion of proof is the subject of the next section—indeed, it is a central theme of this book. Suffice it to say for now that a *proof* is an essay that incontrovertibly shows that a statement is true.

In this section we focus on the notion of a theorem. Reiterating, a *theorem* is a declarative statement about mathematics for which there is a proof.

What is a declarative statement? In everyday English we utter many types of sentences. Some sentences are questions: Where is the newspaper? Other sentences are commands: Come to a complete stop. And perhaps the most common sort of sentence is a *declarative statement*—a sentence that expresses an idea about how something is, such as: It's going to rain tomorrow or The Yankees won last night.

Practitioners of every discipline make declarative statements about their subject matter. The economist says, "If the supply of a commodity decreases, then its price will increase." The physicist asserts, "When an object is dropped near the surface of the earth, it accelerates at a rate of 9.8 meter/sec^2."

Mathematicians also make statements that we believe are true about mathematics. Such statements fall into three categories:

- Statements we know to be true because we can prove them—we call these *theorems*.
- Statements whose truth we cannot ascertain—we call these *conjectures*.
- Statements that are false—we call these *mistakes*!

Please be sure to check your own work for nonsensical sentences. This type of mistake is all too common. Think about every word and symbol you write. Ask yourself, what does this term mean? Do the expressions on the left and right sides of your equations represent objects of the same type?

There is one more category of mathematical statements. Consider the sentence "The square root of a triangle is a circle." Since the operation of extracting a square root applies to numbers, not to geometric figures, the sentence doesn't make sense. We therefore call such statements *nonsense*!

The Nature of Truth

To say that a statement is *true* asserts that the statement is correct and can be trusted. However, the nature of truth is much stricter in mathematics than in any other discipline. For example, consider the following well-known meteorological fact: "In July, the weather in Baltimore is hot and humid." Let me assure you, from personal experience, that this statement is true! Does this mean that every day in every July is hot and humid? No, of course not. It is not reasonable to expect such a rigid interpretation of a general statement about the weather.

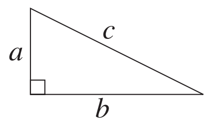
Consider the physicist's statement just presented: "When an object is dropped near the surface of the earth, it accelerates at a rate of 9.8 meter/sec^2." This statement is also true and is expressed with greater precision than our assertion about the climate in Baltimore. But this physics "law" is not absolutely correct. First, the value 9.8 is an approximation. Second, the term *near* is vague. From a galactic perspective, the moon is "near" the earth, but that is not the meaning of *near* that we intend. We can clarify *near* to mean "within 100 meters of the surface of the earth," but this leaves us with a problem. Even at an altitude of 100 meters, gravity is slightly less than at the surface. Worse yet, gravity at the surface is not constant; the gravitational pull at the top of Mount Everest is a bit smaller than the pull at sea level!

Despite these various objections and qualifications, the claim that objects dropped near the surface of the earth accelerate at a rate of 9.8 meter/sec^2 is true. As climatologists or physicists, we learn the limitations of our notion of truth. Most statements are limited in scope, and we learn that their truth is not meant to be considered absolute and universal.

However, in mathematics the word *true* is meant to be considered absolute, unconditional, and without exception.

Let us consider an example. Perhaps the most celebrated theorem in geometry is the following classical result of Pythagoras.

Theorem 4.1 **(Pythagorean)** If a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, then



$$a^2 + b^2 = c^2.$$

The relation $a^2 + b^2 = c^2$ holds for the legs and hypotenuse of every right triangle, absolutely and without exception! We know this because we can prove this theorem (more on proofs later).

Is the Pythagorean Theorem really absolutely true? We might wonder: If we draw a right triangle on a piece of paper and measure the lengths of the sides down to a billionth of an inch, would we have exactly $a^2 + b^2 = c^2$? Probably not, because a drawing of a right triangle is not a right triangle! A drawing is a helpful visual aid for understanding a mathematical concept, but a drawing is just ink on paper. A "real" right triangle exists only in our minds.

On the other hand, consider the statement, "Prime numbers are odd." Is this statement true? No. The number 2 is prime but not odd. Therefore, the statement is false. We might like to say it is nearly true since all prime numbers except 2 are odd. Indeed, there are far more exceptions to the rule "July days in Baltimore are hot and humid" (a sentence regarded to be true) than there are to "Prime numbers are odd."

Mathematicians have adopted the convention that a statement is called *true* provided it is absolutely true without exception. A statement that is not absolutely true in this strict way is called *false*.

An engineer, a physicist, and a mathematician are taking a train ride through Scotland. They happen to notice some black sheep on a hillside.

"Look," shouts the engineer. "Sheep in this part of Scotland are black!"

"Really," retorts the physicist. "You mustn't jump to conclusions. All we can say is that in this part of Scotland there are some black sheep."

"Well, at least on one side," mutters the mathematician.

If-Then

Consider the mathematical and the ordinary usage of the word *prime*. When an economist says that the prime interest rate is now 8%, we are not upset that 8 is not a prime number!

Mathematicians use the English language in a slightly different way than ordinary speakers. We give certain words special meanings that are different from that of standard usage. Mathematicians take standard English words and use them as technical terms. We give words such as *set*, *group*, and *graph* new meanings. We also invent our own words, such as *bijection* and *poset*. (All these words are defined later in this book.)

Not only do mathematicians expropriate nouns and adjectives and give them new meanings, we also subtly change the meaning of common words, such as *or*, for our own purposes.

While we may be guilty of fracturing standard usage, we are highly consistent in how we do it. I call such altered usage of standard English *mathspeak*, and the most important example of mathspeak is the if-then construction.

In the statement "If A, then B," condition A is called the *hypothesis* and condition B is called the *conclusion*.

The vast majority of theorems can be expressed in the form "If A, then B." For example, the theorem "The sum of two even integers is even" can be rephrased "If x and y are even integers, then $x + y$ is also even."

In casual conversation, an if-then statement can have various interpretations. For example, I might say to my daughter, "If you mow the lawn, then I will pay you \$20." If she does the work, she will expect to be paid. She certainly wouldn't object if I gave her \$20 when she didn't mow the lawn, but she wouldn't expect it. Only one consequence is promised.

On the other hand, if I say to my son, "If you don't finish your lima beans, then you won't get dessert," he understands that unless he finishes his vegetables, no sweets will follow. But he also understands that if he does finish his lima beans, then he will get dessert. In this case two consequences are promised: one in the event he finishes his lima beans and one in the event he doesn't.

The mathematical use of if-then is akin to that of "If you mow the lawn, then I will pay you \$20." The statement "If A, then B" means: Every time condition A is true, condition B must be true as well. Consider the sentence "If x and y are even, then $x + y$ is even." All this sentence promises is that when x and y are both even, it must also be the case that $x + y$ is even. (The sentence does not rule out the possibility of $x + y$ being even despite x or y not being even. Indeed, if x and y are both odd, we know that $x + y$ is also even.)

In the statement "If A, then B," we might have condition A true or false, and we might have condition B true or false. Let us summarize this in a chart. If the statement "If A, then B" is true, we have the following.

Condition A	Condition B	
True	True	Possible
True	False	Impossible
False	True	Possible
False	False	Possible

All that is promised is that whenever A is true, B must be true as well. If A is not true, then no claim about B is asserted by "If A, then B."

Here is an example. Imagine I am a politician running for office, and I announce in public, "If I am elected, then I will lower taxes." Under what circumstances would you call me a liar?

- Suppose I am elected and I lower taxes. Certainly you would not call me a liar—I kept my promise.
- Suppose I am elected and I do not lower taxes. Now you have every right to call me a liar—I have broken my promise.
- Suppose I am not elected, but somehow (say, through active lobbying) I manage to get taxes lowered. You certainly would not call me a liar—I have not broken my promise.
- Finally, suppose I am not elected and taxes are not lowered. Again, you would not accuse me of lying—I promised to lower taxes only if I were elected.

The only circumstance under which "If (A) I am elected, then (B) I will lower taxes" is a lie is when A is true and B is false.

In summary, the statement "If A, then B" promises that condition B is true whenever A is true but makes no claim about B when A is false.

Alternative wordings for "If A, then B."

If-then statements pervade all of mathematics. It would be tiresome to use the same phrases over and over in mathematical writing. Consequently, there is an assortment of alternative ways to express "If A, then B." All of the following express exactly the same statement as "If A, then B."

- "A implies B." This can also be expressed in passive voice: "B is implied by A."
- "Whenever A, we have B." Also: "B, whenever A."
- "A is sufficient for B." Also: "A is a sufficient condition for B."
  This is an example of mathspeak. The word *sufficient* can carry, in standard English, the connotation of being "just enough." No such connotation should be ascribed here. The meaning is "Once A is true, then B must be true as well."

- "In order for B to hold, it is enough that we have A."
- "B is necessary for A."

  This is another example of mathspeak. The way to understand this wording is as follows: In order for A to be true, it is *necessarily* the case that B is also true.
- "A, only if B."

  The meaning is that A can happen *only if* B happens as well.
- "$A \implies B$."

  The special arrow symbol $\implies$ is pronounced "implies."
- "$B \impliedby A$".

  The arrow $\impliedby$ is pronounced "is implied by."

If and Only If

The vast majority of theorems are—or can readily be expressed—in the if-then form. Some theorems go one step further; they are of the form "If A then B, and if B then A." For example, we know the following is true:

If an integer x is even, then $x + 1$ is odd, and if $x + 1$ is odd, then x is even.

This statement is verbose. There are concise ways to express statements of the form "A implies B and B implies A" in which we do not have to write out the conditions A and B twice each. The key phrase is *if and only if*. The statement "If A then B, and if B then A" can be rewritten as "A if and only if B." The example just given is more comfortably written as follows:

An integer x is even if and only if $x + 1$ is odd.

What does an if-and-only-if statement mean? Consider the statement "A if and only if B." Conditions A and B may each be either true or false, so there are four possibilities that we can summarize in a chart. If the statement "A if and only if B" is true, we have the following table.

Condition A	Condition B	
True	True	Possible
True	False	Impossible
False	True	Impossible
False	False	Possible

It is impossible for condition A to be true while B is false, because $A \implies B$. Likewise, it is impossible for condition B to be true while A is false, because $B \implies A$. Thus the two conditions A and B must be both true or both false.

Let's revisit the example statement.

An integer x is even if and only if $x + 1$ is odd.

Condition A is "x is even" and condition B is "$x + 1$ is odd." For some integers (e.g., $x = 6$), conditions A and B are both true (6 is even and 7 is odd), but for other integers (e.g., $x = 9$), both conditions A and B are false (9 is not even and 10 is not odd).

Alternative wordings for "A if and only if B."

Just as there are many ways to express an if-then statement, so too are there several ways to express an if-and-only-if statement.

- "A iff B."

  Because the phrase "if and only if" occurs so frequently, the abbreviation "iff" is often used.
- "A is necessary and sufficient for B."
- "A is equivalent to B".

  The reason for the word *equivalent* is that condition A holds under exactly the same circumstances under which condition B holds.
- "A is true exactly when B is true."

  The word *exactly* means that the circumstances under which condition A hold are precisely the same as the circumstances under which B holds.
- "$A \iff B$".

  The symbol $\iff$ is an amalgamation of the symbols $\impliedby$ and $\implies$.

And, Or, and Not

Mathematicians use the words *and*, *or*, and *not* in very precise ways. The mathematical usage of *and* and *not* is essentially the same as that of standard English. The usage of *or* is more idiosyncratic.

Mathematical use of *and*.

The statement "A and B" means that both statements A and B are true. For example, "Every integer whose ones digit is 0 is divisible by 2 *and* by 5." This means that a number that ends in a zero, such as 230, is divisible both by 2 and by 5. The use of *and* can be summarized in the following chart.

A	B	A and B
True	True	True
True	False	False
False	True	False
False	False	False

Mathematical use of *not*.

The statement "not A" is true if and only if A is false. For example, the statement "All primes are odd" is false. Thus the statement "Not all primes are odd" is true. Again, we can summarize the use of *not* in a chart.

A	not A
True	False
False	True

Mathematical use of *or*.

Thus the mathematical usage of *and* and *not* corresponds closely with standard English. The use of *or*, however, does not. In standard English, *or* often suggests a choice of one option or the other, but not both. Consider the question, "Tonight, when we go out for dinner, would you like to have pizza or Chinese food?" The implication is that we'll dine on one or the other, but not both.

In contradistinction, the mathematical *or* allows the possibility of *both*. The statement "A or B" means that A is true, or B is true, or both A and B are true. For example, consider the following:

Suppose x and y are integers with the property that $x|y$ and $y|x$. Then $x = y$ or $x = -y$.

The conclusion of this result says that we may have any one of the following:

- $x = y$ but not $x = -y$ (e.g., take $x = 3$ and $y = 3$).
- $x = -y$ but not $x = y$ (e.g., take $x = -5$ and $y = 5$).
- $x = y$ *and* $x = -y$, which is possible only when $x = 0$ and $y = 0$.

Here is a chart for *or* statements.

A	B	A or B
True	True	True
True	False	True
False	True	True
False	False	False

What Theorems Are Called

The word *theorem* should not be confused with the word *theory*. A *theorem* is a specific statement that can be proved. A *theory* is a broader assembly of ideas on a particular issue.

Some theorems are more important or more interesting than others. There are alternative nouns that mathematicians use in place of *theorem*. Each has a slightly different connotation. The word *theorem* conveys importance and generality. The Pythagorean Theorem certainly deserves to be called a *theorem*. The statement "The square of an even integer is also even" is also a theorem, but perhaps it doesn't deserve such a profound name. And the statement "$6 + 3 = 9$" is technically a theorem but does not merit such a prestigious appellation.

Here we list words that are alternatives to *theorem* and offer a guide to their usage.

Result A modest, generic word for a theorem. There is an air of humility in calling your theorem merely a "result." Both important and unimportant theorems can be called results.

Fact A very minor theorem. The statement "$6 + 3 = 9$" is a fact.

Proposition A minor theorem. A proposition is more important or more general than a fact but not as prestigious as a theorem.

Lemma A theorem whose main purpose is to help prove another, more important theorem. Some theorems have complicated proofs. Often one can break down the job of proving a such theorems into smaller parts. The lemmas are the parts, or tools, used to build the more elaborate proof.

Corollary A result with a short proof whose main step is the use of another, previously proved theorem.

Claim Similar to lemma. A claim is a theorem whose statement usually appears inside the proof of a theorem. The purpose of a claim is to help organize key steps in a proof. Also, the statement of a claim may involve terms that make sense only in the context of the enclosing proof.

Vacuous Truth

What are we to think of an if-then statement in which the hypothesis is impossible? Consider the following.

Statement 4.2 **(Vacuous)** If an integer is both a perfect square and prime, then it is negative.

Is this statement true or false?

The statement is not nonsense. The terms *perfect square* (see Exercise 3.6), *prime*, and *negative* properly apply to integers.

We might be tempted to say that the statement is false because square numbers and prime numbers cannot be negative. However, for a statement of the form "If A, then B" to be declared *false*, we need to find an instance in which clause A is true and clause B is false. In the case of Statement 4.2, condition A is impossible; there are no numbers that are both a perfect square and prime. So we can never find an integer that renders condition A true and condition B false. Therefore, Statement 4.2 is true!

Statements of the form "If A, then B" in which condition A is impossible are called *vacuous*, and mathematicians consider such statements true because they have no exceptions.

Recap

This section introduced the notion of a *theorem*: a declarative statement about mathematics that has a proof. We discussed the absolute nature of the word *true* in mathematics. We examined the if-then and if-and-only-if forms of theorems, as well as alternative language to express such results. We clarified the way in which mathematicians use the words *and*, *or*, and *not*. We presented a number of synonyms for *theorem* and explained their connotations. Finally, we discussed vacuous if-then statements and noted that mathematicians regard such statements as true.

4 Exercises

4.1. Each of the following statements can be recast in the if-then form. Please rewrite each of the following sentences in the form "If A, then B."

- **a.** The product of an odd integer and an even integer is even.
- **b.** The square of an odd integer is odd.
- **c.** The square of a prime number is not prime.
- **d.** The product of two negative integers is negative. (This, of course, is false.)
- **e.** The diagonals of a rhombus are perpendicular.
- **f.** Congruent triangles have the same area.
- **g.** The sum of three consecutive integers is divisible by three.

4.2. Below you will find pairs of statements A and B. For each pair, please indicate which of the following three sentences are true and which are false:

- If A, then B.
- If B, then A.
- A if and only if B.

Note: You do not need to prove your assertions.

a. A: Polygon $PQRS$ is a rectangle. B: Polygon $PQRS$ is a square.
b. A: Polygon $PQRS$ is a rectangle. B: Polygon $PQRS$ is a parallelogram.
c. A: Joe is a grandfather. B: Joe is male.
d. A: Ellen resides in Los Angeles. B: Ellen resides in California.
e. A: This year is divisible by 4. B: This year is a leap year.
f. A: Lines ℓ_1 and ℓ_2 are parallel. B: Lines ℓ_1 and ℓ_2 are perpendicular.

For the remaining items, x and y refer to real numbers.

g. A: $x > 0$. B: $x^2 > 0$.
h. A: $x < 0$. B: $x^3 < 0$.
i. A: $xy = 0$. B: $x = 0$ or $y = 0$.
j. A: $xy = 0$. B: $x = 0$ and $y = 0$.
k. A: $x + y = 0$. B: $x = 0$ and $y = 0$.

The statement "If B, then A" is called the *converse* of the statement "If A, then B."

4.3. It is a common mistake to confuse the following two statements:
a. If A, then B.
b. If B, then A.
Find two conditions A and B such that statement (a) is true but statement (b) is false.

4.4. Consider the two statements
a. If A, then B.
b. (not A) or B.
Under what circumstances are these statements true? When are they false? Explain why these statements are, in essence, identical.

The statement "If (not B), then (not A)" is called the *contrapositive* of the statement "If A, then B."

4.5. Consider the two statements
a. If A, then B.
b. If (not B), then (not A).
Under what circumstances are these statements true? When are they false? Explain why these statements are, in essence, identical.

4.6. Consider the two statements
a. A iff B.
b. (not A) iff (not B).
Under what circumstances are these statements true? Under what circumstances are they false? Explain why these statements are, in essence, identical.

4.7. Consider an equilateral triangle whose side lengths are $a = b = c = 1$. Notice that in this case $a^2 + b^2 \neq c^2$. Explain why this is not a violation of the Pythagorean Theorem.

A side of a spherical triangle is an arc of a great circle of the sphere on which it is drawn.

4.8. Explain how to draw a triangle on the surface of a sphere that has three right angles. Do the legs and hypotenuse of such a right triangle satisfy the condition $a^2 + b^2 = c^2$? Explain why this is not a violation of the Pythagorean Theorem.

4.9. Consider the sentence "A line is the shortest distance between two points." Strictly speaking, this sentence is nonsense.

Find two errors with this sentence and rewrite it properly.

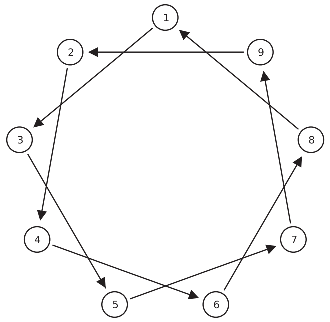
4.10. Consider the following rather grotesque claim: "If you pick a guinea pig up by its tail, then its eyes will pop out." Is this true?

4.11. What are the two plurals of the word *lemma*?

4.12. **More about conjectures.** Where do new theorems come from? They are the creations of mathematicians that begin as *conjectures*: statements about mathematics whose truth we have yet to establish. In other words, conjectures are guesses (usually, educated guesses). By looking at many examples and hunting for patterns, mathematicians express their observations as statements they hope to prove.

The following items are designed to lead you through the process of making conjectures. In each case, try out several examples and attempt to formulate your observations as a theorem to be proved. *You do not have to prove these statements*; for now we simply want you to express what you find in the language of mathematics.

a. What can you say about the sum of consecutive odd numbers starting with 1? That is, evaluate 1, $1+3$, $1+3+5$, $1+3+5+7$, and so on, and formulate a conjecture.
b. What can you say about the sum of consecutive perfect cubes, starting with 1. That is, what can you say about 1^3, 1^3+3^3, $1^3+3^3+5^3$, $1^3+3^3+5^3+7^3$, and so on.



c. Let n be a positive integer. Draw n lines (no two of which are parallel) in the plane. How many regions are formed?

d. Place n points evenly around a circle. Starting at one point, draw a path to every other point around the circle until returning to start. In some instances, every point is visited and in some instances some are missed. Under what circumstances is every point visited (as in the figure with $n = 9$)?
Suppose instead of jumping to every second point, we jump to every third point. For what values of n does the path touch every point?
Finally, suppose we visit every k^{th} point (where k is between 1 and n). When does the path touch every point?

e. A school has a long hallway of lockers numbered 1, 2, 3, and so on up to 1000. In this problem we'll refer to *flipping* a locker to mean opening a closed locker or closing an open locker. That is, to *flip* a locker is to change its closed/open state.

- Student #1 walks down the hallway and closes all the lockers.
- Student #2 walks down the hallway and flips all the even numbered lockers. So now, the odd lockers are closed and the even lockers are open.
- Student #3 walks down the hall and flips all the lockers that are divisible by 3.
- Student #4 walks down the hall and flips all the lockers that are divisible by 4.
- Likewise students 5, 6, 7, and so on walk down the hall in turn, each flipping lockers divisible by their own number until finally student 1000 flips the (one and only) locker divisible by 1000 (the last locker).

Which lockers are open and which are closed? Generalize to any number of lockers. Note: We ask you to prove your conjecture later; see Exercise 24.19.

5 Proof

We create mathematical concepts via definitions. We then posit assertions about mathematical notions, and then we try to prove our ideas are correct.

What is a *proof*?

In science, truth is borne out through experimentation. In law, truth is ascertained by a trial and decided by a judge and/or jury. In sports, the truth is the ruling of referees to the best of their ability. In mathematics, we have *proof*.

Truth in mathematics is not demonstrated through experimentation. This is not to say that experimentation is irrelevant for mathematics—quite the contrary! Trying out ideas and examples helps us to formulate statements we believe to be true (conjectures); we then try to prove these statements (thereby converting conjectures to theorems).

For example, recall the statement "All prime numbers are odd." If we start listing the prime numbers beginning with 3, we find hundreds and thousands of prime numbers, and they are all odd! Does this mean all prime numbers are odd? Of course not! We simply missed the number 2.

Let us consider a far less obvious example.

Conjecture 5.1 **(Goldbach)** Every even integer greater than two is the sum of two primes.

Let's see that this statement is true for the first few even numbers. We have

$$4 = 2 + 2 \qquad 6 = 3 + 3 \qquad 8 = 3 + 5 \qquad 10 = 3 + 7$$
$$12 = 5 + 7 \qquad 14 = 7 + 7 \qquad 16 = 11 + 5 \qquad 18 = 11 + 7.$$

Mathspeak! A proof is often called an *argument*. In standard English, the word *argument* carries a connotation of disagreement or controversy. No such negative connotation should be associated with a mathematical argument. Indeed, mathematicians are honored when their proofs are called "beautiful arguments."

One could write a computer program to verify that the first few billion even numbers (starting with 4) are each the sum of two primes. Does this imply Goldbach's Conjecture is true? No! The numerical evidence makes the conjecture believable, but it does not prove that it is true. To date, no proof has been found for Goldbach's Conjecture, so we simply do not know whether it is true or false.

A proof is an essay that incontrovertibly shows that a statement is true. Mathematical proofs are highly structured and are written in a rather stylized manner. Certain key phrases