Problem 1. Suppose I saw a parade of elephants the other day and I told you that every time I saw a pink elephant, the next elephant was also pink.

Which of the following statements can you conclude are true?

A) At least one elephant in the parade was pink.

B) Every elephant in the parade was pink.

C) The first elephant in the parade was not pink.

D) More than one of the above

E) None of the above

Problem 2. Suppose that we know S is a subset of $\mathbb{Z}$, that if $x \in S$ then $x + 1 \in S$ as well, and that $4 \in S$. Which of the following are possible?

A) $S = \{4\}$

B) $S = \mathbb{N}$

C) $S = \mathbb{Z}$

D) More than one of the above

E) None of the above

Problem 3. Prove that $1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$ for all positive integers n. (Notice that this formula can be written $\sum_{m=0}^{n} \binom{m}{1} = \binom{n+1}{2}$.)

Problem 4. The *triangular numbers* are the numbers $T_n = \binom{n}{2}$, where $n \geq 0$.

(i) How many triangular numbers are there between 1 and 10?

A) 1 B) 4 C) 7 D) 10

(ii) Find a formula for the sum $T_1 + T_2 + \cdots + T_n$ of the first n triangular numbers and prove it using induction.

Solution. We have to guess a formula, somehow. Let's try $\binom{n+1}{3}$. This formula works when $n = 1$ since $T_1 = \binom{1}{2} = 0$ and $\binom{1}{2} + \binom{1}{3} = 0$. Now we assume the formula holds for n and prove it for $n + 1$. We have

$$T_1 + T_2 + \cdots + T_n + T_{n+1} = \binom{n+1}{3} + \binom{n+1}{2} = \binom{n+1}{3}$$

by application of the general formula, $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$. Therefore if the formula holds for n it also holds for $n + 1$. By induction, we conclude that it holds for all n. $\square$

Problem 5. Let n be a positive integer. For every positive integer m there is exactly one integer r with $0 \leq r < n$ such that $m \equiv r \pmod{n}$.

A) True B) False

Problem 6. The division algorithm says that if n is a positive integer then every other integer m can be written as $qn + r$ where q and r are integers and $0 \leq r < n$. Prove the division algorithm by induction.

Solution. We work by induction on m. If $m = 0$ then we may take $q = r = 0$ and get $0 = 0n + 0$. This is the base case. For the induction step, we assume the division algorithm holds for m and prove it for $m + 1$. Then we can write $m = qn + r$ with $0 \leq r < n$. We separate two cases, depending on whether $r = n - 1$. If $r \neq n - 1$ then $m + 1 = qn + (r + 1)$ proves the division algorithm in this case. If $r = n - 1$ then $m + 1 = qn + n = (q + 1)n + 0$, which proves the division algorithm for $m + 1$ in the other case.

This proves the division algorithm for all $m \geq 0$. If $m < 0$ then $-m > 0$ so we can write $-m = qn + r$ for some q and $0 \leq r < n$. Then $m = -qn - r = (-q - 1)n + (n - r)$ is an expression for m in the required form $\square$

Solution. We use strong induction on m. First we prove it for all $0 \leq m < n$. In this case, we can take $r = m$ and $q = 0$, for $m = 0n + m$. Now assume that the division algorithm holds for all $0 \leq m < m'$. We prove it for m'. We can assume that $m' \geq n$. Then $0 \leq m' - n < m'$ so we can apply the division algorithm to $m' - n$ and find that it is possible to write

$$m' - n = qn + r$$

for integers q and r with $0 \leq r < n$. But then

$$m' = qn + n + r = (q + 1)n + r$$

and $q + 1$ and r are integers with $0 \leq r < n$, so the division algorithm holds for m' as well. $\square$

Problem 7. Prove the binomial theorem $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ using induction.

Problem 8. Find a formula for the sum of the first n consecutive cubes

$$1 + 2^3 + 3^3 + 4^3 + \cdots + n^3$$

and prove it by induction.

Solution. The formula is $\sum_{k=1}^{n-1} k^3 = \binom{n}{2}^2$. This formula holds for $n = 0$, for $\binom{0}{2} = 0$ and $\sum_{k=1}^{0} k^3 = 0$ as well. Assuming this formula holds for n, we prove it for $n + 1$. We have

$$\sum_{k=1}^{n} k^3 = \sum_{k=1}^{n-1} k^3 + (n+1)^3 = \binom{n}{2}^2 + n^3.$$

Recall that $\binom{n}{2} = \frac{n(n-1)}{2}$ so we may expand this into

$$\begin{aligned} \frac{n^2(n-1)^2}{4} + n^3 &= \frac{n^4 - 2n^2 + 1}{4} + n^3 \\ &= \frac{n^4 + 2n^2 + 1}{4} \\ &= \frac{(n+1)^2 n^2}{4} = \binom{n+1}{2}^2. \end{aligned}$$

By induction, we conclude that the formula holds for all n. $\square$