Problem 1. Suppose that $P(n)$ is a sentence that depends on an integer n. If you can prove that, for all integers n, the sentence $P(n)$ implies $P(n+1)$ then you can deduce that $P(n)$ is true for all values of n.

A) True B) False

Problem 2. Every nonempty set of positive rational numbers has a least element.

A) True B) False

Problem 3. Let n be a positive integer. For every positive integer m there is exactly one integer r with $0 \leq r < n$ such that $m \equiv r \pmod{n}$.

A) True B) False

Problem 4. The division algorithm says that if n is a positive integer then every other integer m can be written as $qn + r$ where q and r are integers and $0 \leq r < n$. Prove the division algorithm by induction.

Solution. We work by induction on m. If $m = 0$ then we may take $q = r = 0$ and get $0 = 0n + 0$. This is the base case. For the induction step, we assume the division algorithm holds for m and prove it for $m + 1$. Then we can write $m = qn + r$ with $0 \leq r < n$. We separate two cases, depending on whether $r = n - 1$. If $r \neq n - 1$ then $m + 1 = qn + (r + 1)$ proves the division algorithm in this case. If $r = n - 1$ then $m + 1 = qn + n = (q + 1)n + 0$, which proves the division algorithm for $m + 1$ in the other case.

This proves the division algorithm for all $m \geq 0$. If $m < 0$ then $-m > 0$ so we can write $-m = qn + r$ for some q and $0 \leq r < n$. Then $m = -qn - r = (-q - 1)n + (n - r)$ is an expression for m in the required form □

Solution. We use strong induction on m. First we prove it for all $0 \leq m < n$. In this case, we can take $r = m$ and $q = 0$, for $m = 0n + m$. Now assume that the division algorithm holds for all $0 \leq m < m'$. We prove it for m'. We can assume that $m' \geq n$. Then $0 \leq m' - n < m'$ so we can apply the division algorithm to $m' - n$ and find that it is possible to write

$$m' - n = qn + r$$

for integers q and r with $0 \leq r < n$. But then

$$m' = qn + n + r = (q + 1)n + r$$

and $q + 1$ and r are integers with $0 \leq r < n$, so the division algorithm holds for m' as well. □

Problem 5. Prove the binomial theorem $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ using induction.

Problem 6. Find a formula for the first n consecutive cubes

$$1 + 2^3 + 3^3 + 4^3 + \cdots + n^3$$

and prove it by induction.

Solution. The formula is $\sum_{k=1}^{n-1} k^3 = \binom{n}{2}^2$. This formula holds for $n = 0$, for $\binom{0}{2} = 0$ and $\sum_{k=1}^{0} k^3 = 0$ as well. Assuming this formula holds for n, we prove it for $n+1$. We have

$$\sum_{k=1}^{n} k^3 = \sum_{k=1}^{n-1} k^3 + (n+1)^3 = \binom{n}{2}^2 + n^3.$$

Recall that $\binom{n}{2} = \frac{n(n-1)}{2}$ so we may expand this into

$$\begin{aligned}
\frac{n^2(n-1)^2}{4} + n^3 &= \frac{n^4 - 2n^2 + 1}{4} + n^3 \\
&= \frac{n^4 + 2n^2 + 1}{4} \\
&= \frac{(n+1)^2 n^2}{4} = \binom{n+1}{2}^2.
\end{aligned}$$

By induction, we conclude that the formula holds for all n. $\square$