Problem 1. Prove by contradiction that the sum of a rational number and an irrational number cannot be rational.

Problem 2. How many ways are there to rearrange the list $(1, 2, 3, 4, 5)$ such that each of 1, 3, and 5 does not wind up in the same place?

A) 2
B) 45
C) 48
D) 64
E) 120

Solution. The answer is 64. Let S be the set of rearrangements of the list $(1, 2, 3, 4, 5)$. This contains $5! = 120$ elements. Let A_i be the subset of rearrangements that hold i in place. Then we want to count $|S - A_1 - A_3 - A_5|$. We can try to compute this number as

$$|S| - |A_1| - |A_3| - |A_5|$$

but then we will have subtracted everything in $A_1 \cap A_3$, $A_1 \cap A_5$ and $A_3 \cap A_5$ twice. We can try adding those back in:

$$|S| - |A_1| - |A_3| - |A_5| + |A_1 \cap A_3| + |A_1 \cap A_5| + |A_1 \cap A_3|$$

Now if $x \in A_1 \cap A_3$ and $x \notin A_5$ it will be counted

$$1 - 1 - 1 + 1 = 0$$

times, but if $x \in A_1 \cap A_3 \cap A_5$, it will be counted

$$1 - 1 - 1 - 1 + 1 + 1 + 1 = 1$$

times. To correct for this, we subtract off $|A_1 \cap A_3 \cap A_5|$ and get

$$|S| - |A_1| - |A_3| - |A_5| + |A_1 \cap A_3| + |A_1 \cap A_5| + |A_3 \cap A_5| - |A_1 \cap A_3 \cap A_5|.$$

To conclude, we should compute these values. The size of A_i is $4!$; the size of $A_i \cap A_j$ is $3!$ if $i \neq j$. And the size of $A_1 \cap A_3 \cap A_5$ is $2! = 2$. Therefore the number of rearrangemenets that move 1 and 3 and 5 is

$$5! - 3 \times 4! - 3 \times 3! + 2! = 64.$$

$\square$

Problem 3. Suppose that S is a set and that for every $s \in S$ we have a set A_s. For every subset $T \subset S$, let us define

$$A_T = \bigcap_{t \in T} A_t.$$

Suppose that $s \in S$. Express, in terms of $|S|$ and the ordinary operations of arithmetic, the number of $T \subset S$ such that $s \in A_T$. How many of these T have an even and how many have an odd number of elements?

Solution. The number is $2^{|S|-1}$ if $S \neq \varnothing$ and it is 0 if $S = \varnothing$. The numbers with even and odd sizes are both $2^{|S|-2}$ if $|S| \geq 2$. If $|S| = 1$ there is one odd-sized T and no even-sized T-s. If $|S| = 0$ then the numbers of odd- and even-sized T-s are both zero. $\square$

Problem 4. Find a statement of the inclusion-exclusion principle for an arbitrary number of sets that does not rely on ellipses.

Problem 5. Prove that

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.$$

Solution. How many ways are there to partition a set S with n elements into three subsets A, B, and C with sizes $|A| = a$, $|B| = b$, and $|C| = c$. Let's assume that $a + b + c = n$. Let's write $a = m$ and $k = a + b$. We can first choose $A \cup B$ from S, for which there are $\binom{n}{k}$ possibilities, and then we can choose A from $A \cup B$, for which there are $\binom{k}{m}$ possibilities. This gives a total of $\binom{n}{k}\binom{k}{m}$ ways.

Or we could first choose A, for which there are $\binom{n}{m}$ possibilities and then choose B from $S - A$. Since $|S - A| = n - m$ and $|B| = k - m$, there are $\binom{n-m}{k-m}$ possibilities for B once A has been chosen. This makes a total of $\binom{n}{m}\binom{n-m}{k-m}$ possibilities. $\square$

Problem 6. (i) Prove that a product of odd numbers is odd.

(ii) Prove that if x is an integer such that $2|x^2$ then $2|x$.

Problem 7. A number is called *rational* if it can be expressed as u/v where u and v are integers (and $v \neq 0$). Prove by contradiction that $\sqrt{2}$ is not rational. You may use the following fact about the number 2: If x is an integer and $2|x^2$ then $2|x$. (Hint: Choose u to be the smallest positive integer such that there is a positive integer v with $u/v = \sqrt{2}$ and find a contradiction.)

Solution. Choose u and v as in the hint. If $\sqrt{2} = u/v$ then $u^2 = 2v^2$. Then $2|u^2$ so $2|u$. But this means $u = 2w$ for some positive integer w so that $2w^2 = v^2$. Then $2|v^2$ so $2|v$ so $v = 2x$ for some positive integer x. Therefore $w^2 = 2v^2$ so $w/v = \sqrt{2}$. But $w < u$ so this contradicts the assumption that u was the smallest positive integer such that $\sqrt{2} = u/v$ for some positive integer v. $\square$

Problem 8. Use a proof by contradiction to prove that there are infinitely many prime numbers.