Problem 1. How many of the integers x with $1 \leq x \leq 120$ are divisible by at least one of 2 or 5?

A) 12 B) 60 C) 72 D) 84 E) 96

Solution. The correct answer is 72. □

Problem 2. How many of the integers x with $1 \leq x \leq 120$ are divisible by at least one of 2, 3, or 5?

A) 4 B) 40 C) 76 D) 108 E) 112

Solution. The correct answer is 88. □

Problem 3. How many ways are there to rearrange the list $(1, 2, 3, 4, 5)$ such that 1, 3, and 5 do not wind up in the same place?

A) $2! = 2$

B) $5! - 5 \times 4! + 10 \times 3! - 10 \times 2! + 5 \times 1! = 120 - 120 + 60 - 20 + 5 = 45$

C) $5! - 3 \times 4! = 120 - 72 = 48$

D) $5! - 3 \times 4! + 3 \times 3! - 2! = 120 - 72 + 18 - 2 = 64$

E) $5! = 120$

Solution. The answer is 64. Let S be the set of rearrangements of the list $(1, 2, 3, 4, 5)$. This contains $5! = 120$ elements. Let A_i be the subset of rearrangements that hold i in place. Then we want to count $|S - A_1 - A_3 - A_5|$. We can try to compute this number as

$$|S| - |A_1| - |A_3| - |A_5|$$

but then we will have subtracted everything in $A_1 \cap A_3$, $A_1 \cap A_5$ and $A_3 \cap A_5$ twice. We can try adding those back in:

$$|S| - |A_1| - |A_3| - |A_5| + |A_1 \cap A_3| + |A_1 \cap A_5| + |A_1 \cap A_3|$$

Now if $x \in A_1 \cap A_3$ and $x \notin A_5$ it will be counted

$$1 - 1 - 1 + 1 = 0$$

times, but if $x \in A_1 \cap A_3 \cap A_5$, it will be counted

$$1 - 1 - 1 - 1 + 1 + 1 + 1 = 1$$

times. To correct for this, we subtract off $|A_1 \cap A_3 \cap A_5|$ and get

$$|S| - |A_1| - |A_3| - |A_5| + |A_1 \cap A_3| + |A_1 \cap A_5| + |A_3 \cap A_5| - |A_1 \cap A_3 \cap A_5|.$$

To conclude, we should compute these values. The size of A_i is $4!$; the size of $A_i \cap A_j$ is $3!$ if $i \neq j$. And the size of $A_1 \cap A_3 \cap A_5$ is $2! = 2$. Therefore the number of rearrangemenets that move 1 and 3 and 5 is

$$5! - 3 \times 4! - 3 \times 3! + 2! = 64.$$

□

Problem 4. Let S be a set with three subsets A, B, and C. Suppose that x is an element of S. How many of the sets

$$A, B, C, A \cap B, A \cap C, B \cap C, A \cap B \cap C$$

could contain x? Answer as precisely as possible.

A) 1
B) 1 or 7
C) 1, 3, or 7
D) an odd number
E) Any number between 1 and 7

Solution. The correct answer is 1, 3 or 7. If x is in just one of A, B, or C then the number is 1. If x is in two of them then the answer is 3. If x is in all three then the answer is 7. □

Problem 5. Suppose that S is a set and that for every $s \in S$ we have a set A_s. For every subset $T \subset S$, let us define

$$A_T = \bigcap_{t \in T} A_t.$$

Suppose that $s \in S$. Express, in terms of $|S|$ and the ordinary operations of arithmetic, the number of $T \subset S$ such that $s \in A_T$. How many of these T have an even and how many have an odd number of elements?

Solution. The number is $2^{|S|-1}$ if $S \neq \varnothing$ and it is 0 if $S = \varnothing$. The numbers with even and odd sizes are both $2^{|S|-2}$ if $|S| \geq 2$. If $|S| = 1$ there is one odd-sized T and no even-sized T-s. If $|S| = 0$ then the numbers of odd- and even-sized T-s are both zero. □