Problem 1. Let n be a positive integer. Compute

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n}.$$

Solution. The answer is 0. One way to see this is to remember that $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$ and substitute $x = 1$ and $y = -1$.

Here is a combinatorial proof. Rearrange the equation into

$$\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots.$$

The left side is the number of subsets of an n-element set S with an even number of elements. The right side is the number of subsets with an odd number of elements. We can construct a one-to-one correspodence between these two collections.

Let's pick an element of S—call it x. If we have a set with an even number of elements, either it contains x or it doesn't. If it contains x then remove x from it to get a set with an odd number of elements. If it does not contain x then add x to it to get a set with an odd number of elements. The same procedure can be used to convert a set with an odd number of elements into a set with an even number of elements. These procedures reverse one another, so this gives a one-to-one correspondence between the set of even-size subsets and the set of odd-size subsets of S. □

Problem 2. Let S be a set with n elements. How many ways are there to partition S into *two* subsets.

A) 1 B) n C) $2^n - 2$ D) 2^n E) None of these

Solution. Let's first consider all lists (A, B) where $\{A, B\}$ is a partition of S. We have $B = S - A$, so the number of such lists is equal to the number of subsets of S excluding $A = \varnothing$ and $A = S$ (the latter exclusion because B cannot be empty). The number of these is $2^n - 2$ if $n > 0$ and is 0 if $n = 0$.

We aren't done, because we have counted every partition twice! Consider the "defines the same partition as" equivalence relation on lists (A, B) as above. Each equivalence class has two elements, (A, B) and (B, A). Therefore the total number of partitions of S with two parts is $\frac{2^n-2}{2} = 2^{n-1} - 1$ if $n \geq 1$ and is 0 if $n = 0$. □

Problem 3. Let S be a set with n elements and let a and b be positive integers such that $a + b = n$. How many ways are there to partition S into two subsets of sizes a and b?

A) ab B) $\frac{1}{2}\binom{n}{a}$ C) $\binom{n}{b}$ D) 2^n E) None of these

Solution. The answer is $\binom{n}{a} = \binom{n}{b}$ if $a \neq b$ and $\frac{1}{2}\binom{n}{a}$ if $a = b$. Here is a combinatorial proof: Suppose first that $a \neq b$. We show that in this situation, there is a one-to-one correspondence between partitions of S into subsets of sizes a and b and subsets of S of size a. Indeed, if P is a partition of S into subsets of sizes a and b then there is exactly one $A \in P$ whose size is a. Thus P corresponds to the subset A of size a; conversely, if we had a subset $A \subset S$ of size a then $P = \{A, S - A\}$ is a partition of S into subsets of sizes a and b.

This works fine if $a \neq b$, but it doesn't work if $a = b$. Define an equivalence relation on the set of subsets of S of size a by saying $A \equiv A$ and $A \equiv S - A$ for all $A \subset S$. Each partition of S into two subsets of size a corresponds to one equivalence class, and each equivalence class has two elements. Therefore the number of partitions of S into two sets of size A is $\frac{1}{2}\binom{n}{a}$. □

Problem 4. Let S be a set with n elements and let a, b, and c be three positive integers with $a + b + c = n$. You may assume that a, b, and c are all different numbers. Devise a formula using addition, subtraction, multiplication, division, exponentiation, and the factorial for the number of partitions of S into three subsets of sizes a, b, and c.

Solution. $\binom{n}{a}\binom{n-a}{b} = \frac{n!}{a!b!c!}$ □

Problem 5. How does your formula from the last problem change when $a = b \neq c$? What about when $a = b = c$?