Problem 1. Is $\{\{1\},\{2,3\},\{5\}\}$ a partition?
A) Yes B) No C) Of what?

Problem 2. How many partitions are there of the empty set?
A) 0 B) 1 C) ∞ D) The answer is not defined.

Solution. The empty partition is the only one. $\square$

Problem 3. How many distinct rearrangements are there of the letters of my name, JONATHAN?
A) 1 B) 8 C) $2 \times 7!$ D) $8!$ E) None of these

Solution. The solution is $8!/4 = 2 \times 7!$. $\square$

Problem 4. Let n be a positive integer. Compute

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n}.$$

A) $\binom{n}{n/2}$ B) $-\binom{n}{n/2}$ C) 0 D) Depends on n.

Solution. The answer is 0. One way to see this is to remember that $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$ and substitute $x = 1$ and $y = -1$.

Here is a combinatorial proof. Rearrange the equation into

$$\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots.$$

The left side is the number of subsets of an n-element set S with an even number of elements. The right side is the number of subsets with an odd number of elements. We can construct a one-to-one correspodence between these two collections.

Let's pick an element of S—call it x. If we have a set with an even number of elements, either it contains x or it doesn't. If it contains x then remove x from it to get a set with an odd number of elements. If it does not contain x then add x to it to get a set with an odd number of elements. The same procedure can be used to convert a set with an odd number of elements into a set with an even number of elements. These procedures reverse one another, so this gives a one-to-one correspondence between the set of even-size subsets and the set of odd-size subsets of S. $\square$

Problem 5. Let S be a finite set. Which is greater?

A) The number of partitions of S.

B) The number of equivalence relations on S.

C) They are equal.

D) The answer depends on S.

Solution. The two numbers are equal. $\square$

Problem 6. Let S be a set with n elements. How many ways are there to partition S into *two* subsets.

A) 1 B) n C) $2^n - 2$ D) 2^n E) None of these

Solution. Let's first consider all lists (A, B) where $\{A, B\}$ is a partition of S. We have $B = S - A$, so the number of such lists is equal to the number of subsets of S excluding $A = \varnothing$ and $A = S$ (the latter exclusion because B cannot be empty). The number of these is $2^n - 2$ if $n > 0$ and is 0 if $n = 0$.

We aren't done, because we have counted every partition twice! Consider the "defines the same partition as" equivalence relation on lists (A, B) as above. Each equivalence class has two elements, (A, B) and (B, A). Therefore the total number of partitions of S with two parts is $\frac{2^n-2}{2} = 2^{n-1} - 1$ if $n \geq 1$ and is 0 if $n = 0$. $\square$

Problem 7. Let S be a set with n elements and let a, b, and c be three positive integers with $a + b + c = n$. You may assume that a, b, and c are all different numbers. Devise a formula using addition, subtraction, multiplication, division, exponentiation, and the factorial for the number of partitions of S into three subsets of sizes a, b, and c.

Solution. $\binom{n}{a}\binom{n-a}{b} = \frac{n!}{a!b!c!}$ $\square$

Problem 8. How does your formula from the last problem change when $a = b \neq c$? What about when $a = b = c$?