Problem 1. Draw a picture of congruence modulo 4 as a relation on the integers.

Problem 2. Every relation on the empty set is an equivalence relation.
A) True B) False

Problem 3. Suppose that Q and R are equivalence relations on a set S. Is $Q \cup R$ an equivalence relation?
A) Yes B) No C) Sometimes D) I don't know

Solution. Sometimes, but usually it's not. If $Q = R$, then of course $Q \cup R = R$ is an equivalence relation. On the other hand, if Q is congruence modulo 2 and R is congruence modulo 3, then $(a, b) \in Q \cup R$ if $a - b$ is divisible by 2 or by 3. Therefore $(0, 2) \in Q \cup R$ and $(2, 5) \in Q \cup R$ but $(0, 5) \notin Q \cup R$. $\square$

Problem 4. Let Q and R be two equivalence relations on a set S. Is $Q \cap R$ an equivalence relation on S?
A) Yes B) No C) Sometimes D) I don't know

Solution. Yes, it is. We have to check it is reflexive, symmetric, and transitive. For reflexivity, note that for any $x \in S$ we have $(x, x) \in Q$ and $(x, x) \in R$ since both Q and R are reflexive. Therefore $(x, x) \in Q \cap R$ for any $x \in S$. Thus $Q \cap R$ is reflexive.

For symmetric, consider a pair of elements x and y of S. We have to show that whenever (x, y) is in $Q \cap R$ so is (y, x). If $(x, y) \in Q \cap R$ then $(x, y) \in Q$ and $(x, y) \in R$, by definition of intersection. But both Q and R are symmetric relations, so this implies that $(y, x) \in Q$ and $(y, x) \in R$. Thus $(y, x) \in Q \cap R$, again by the definition of intersection of sets.

Finally we prove symmetry. We have to check that whenever $(x, y) \in Q \cap R$ and $(y, z) \in Q \cap R$, we also have $(x, z) \in Q \cap R$. Suppose that $(x, y) \in Q \cap R$ and $(y, z) \in Q \cap R$. Then $(x, y) \in Q$ and $(y, z) \in Q$ and $(x, y) \in R$ and $(y, z) \in R$. As both Q and R are transitive relations, this means that $(x, z) \in Q$ and $(x, z) \in R$. Thus $(x, z) \in Q \cap R$. This demonstrates that $Q \cap R$ is transitive.

Altogether, we have shown that $Q \cap R$ is reflexive, symmetric, and transitive. It is therefore an equivalence relation. $\square$

Problem 5. Suppose that Q is congruence modulo 2 and R is congruence modulo 3. Find a familiar name for the relation $Q \cap R$.

Solution. It is congruence modulo 6. $\square$

Problem 6. Let n be an integer. Suppose that $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$. Prove that $a + b \equiv a' + b' \pmod{n}$.

Solution. As $a \equiv a' \pmod{n}$ we have $a - a' = kn$ for some integer k. Likewise, we have $b - b' = \ell n$ for some integer ℓ. Therefore $(a + b) - (a' + b') = (a - a') + (b - b') = kn + \ell n = (k + \ell)n$ is an integer multiple of n. Thus $a + b \equiv a' + b' \pmod{n}$. $\square$

Problem 7. Suppose that R is a relation on a set S. Let $Q = S \times S - R$. Prove that R is reflexive if and only if Q is irreflexive.