Problem 1. Let S be a set. How many relations are there on S that are both equivalence relations and functions? Justify your answer.

Problem 2. Compute the sum of all numbers between 0 and 100 that are not divisible by 2 or 5.

Problem 3. Define a set Y as follows:

$$Y = \{X \,:\, X \text{ is a set and } X \notin X\}$$

Is Y an element of itself?

Problem 4. Let S be a set and let $P(x)$ be a sentence that depends on the choice of an element x of S. Which of the following sentences implies the other? Justify your answer.

(a) $\forall x \in S, P(x)$

(b) $\exists x \in S, P(x)$

(c) They are logically equivalent

(d) Neither implies the other

Problem 5. Determine whether the following sentence is true or false and justify your answer: For every two sets A and B there is a function with domain A and codomain B that is not a bijection.

Problem 6. Let $S = \mathbb{R} \times \mathbb{R}$. Let

$$R = \{((a, b), (c, d)) \in S \,:\, ad - bc = 0\}.$$

Is this relation an equivalence relation? Say which properties of an equivalence relation hold and which fail and justify your assertions.

Problem 7. Suppose that A and B are finite sets and that there is at least one bijection from A to B. Which of the following are possible?

(i) There is a function from A to B that is neither injective nor surjective.

(ii) There is a function from A to B that is injective but not surjective.

(iii) There is a function from A to B that is surjective but not injective.

What if A and B are infinite?

Problem 8. Let S be a set. Which is larger?

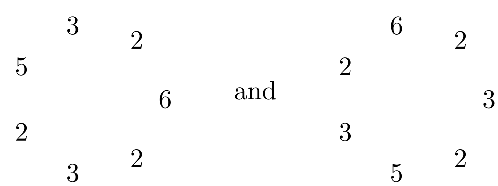
A) The number of equivalence relations on S

B) The number of partitions of S

C) They are the same size

D) The question is not well-defined

Problem 9. A *cyclic list* is a collection of objects arranged in a circle. A rotation of a cyclic list is still the same list. For example,



are representations of the same cyclic list of size 7, but

$$\begin{array}{cccc} & 5 & 2 & \\ 3 & & & \\ & & & 3 \\ 2 & & & \\ & 6 & 2 & \end{array}$$

is a different cyclic list. Compute the number of cyclic lists of size k drawn from a pool of n elements.

How many cyclic lists are there of size 7 drawn from 6 elements?
A) 7^5 B) 7^6 C) 6^7 D) 6^6 E) None of these

Problem 10. Suppose that A and B are finite sets and there are injections $f : A \to B$ and $g : B \to A$. Which of the following can you conclude?
A) $|A| \leq |B|$ B) $|A| \geq |B|$ C) $|A| = |B|$ D) None of these

Problem 11. Let n be an integer. For an integer a, let $[a]$ be the equivalence class of a modulo n. Let S be the set of all equivalence classes of integers modulo n. Prove that the functions

$$\alpha : S \times S \to S$$
$$\mu : S \times S \to S$$

defined by $\alpha([a], [b]) = [a + b]$ and $\mu([a], [b]) = [ab]$ are well defined.

Problem 12. In this problem, the elements of sets are only allowed to be other sets. For any set S, we define the *depth* of S to be the length of the longest chain

$$x_1 \in x_2 \in \cdots \in x_n \in S.$$

(a) List all sets of depths zero, one, and two.

(b) Prove that, for any positive integer n, the number of sets of depth n is equal to 2^m where m is the number of sets of depth $< n$.

(c) How many sets are there of depth 4?