Problem 1. Prove that $2^n = \sum_{k=0}^{n} \binom{n}{k}$.

Problem 2. Let S be a set with n elements. Write $\begin{pmatrix} n \\ k \\ \ell \end{pmatrix}$ for the number of ways of choosing a pair (T, U) where $U \subset T \subset S$ and $|U| = \ell$ and $|T| = k$. Find a formula for $\begin{pmatrix} n \\ k \\ \ell \end{pmatrix}$ using addition, subtraction, multiplication, division, and the factorial.

Problem 3. (i) Prove that an integer n is divisible by both 2 and 5 if and only if it is divisible by 10.

(ii) Using the previous part, compute the sum of all numbers between 1 and 100 (inclusive) that are divisible by either 2 or 5.

Problem 4. Let n be a positive integer. Show that n can be written a sum of distinct powers of 2.

Problem 5. Suppose that $P(n)$ is a sentence that depends on an integer n. Assume that the following two sentences are true:

(I) $\exists k,\ P(k)$

(II) $\forall n,\ P(n) \iff P(n+1)$

Prove that $P(n)$ is true for all $n \in \mathbb{Z}$. You may use strong or weak induction or proof by smallest counterexample, but the other variants of induction discussed in class and on the homework are not allowed.

Problem 6. Suppose that a is a sequence of numbers with

$$\begin{aligned} a_0 &= 2 \\ a_1 &= 3 \\ a_n &= 3a_{n-1} - 2a_{n-2}. \end{aligned}$$

Prove that $a_n = 2^n + 1$ for all n.