Math 2001-002 Spring 2014
Homework 24

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Problem 1. Let $P(n)$ be a sentence that depends on an integer n. Let m be a positive integer. Suppose the following two statements are true:

(I) If, for some integer k, the sentences $P(k), P(k+1), \ldots, P(k+m)$ are all true then $P(k-1)$ and $P(k+m+1)$ are both true.

(II) There is some integer k such that $P(k), P(k+1), \ldots, P(k+m)$ are all true.

Deduce using induction or proof by smallest counterexample that $P(m)$ is true for all m.

Solution. Let k be an integer such that $P(k), P(k+1), \ldots, P(k+m)$ all hold. We prove by induction that $P(n)$ holds for all $n \geq k$. Let $Q(\ell)$ be the sentence "$P(k+\ell), P(k+\ell+1), \ldots, P(k+\ell+m)$ all hold". By assumption, $Q(0)$ is true. Now we prove that $Q(\ell)$ is true provided that $Q(\ell-1)$. Indeed, if $P(k+\ell-1), \ldots, P(k+\ell-1+m)$ all hold then $P(k+\ell+m)$ is also true by Property (I). Therefore $P(k+\ell), P(k+\ell+1), \ldots, P(k+\ell+m)$ are all true, so $Q(k+\ell)$ is true. This proves the inductive step, and therefore shows that $P(n)$ holds for all $n \geq k$.

Now let $R(\ell)$ be the sentence "$P(k-\ell), P(k-\ell+1), \ldots, P(k-\ell+m)$ are all true". Then $R(0)$ is true by assumption. We prove that $R(\ell-1)$ implies $R(\ell)$. Indeed, $R(\ell-1)$ tells us that $P(k-\ell+1), P(k-\ell+2), \ldots, P(k-\ell+m+1)$ are all true. Therefore, by Property (I), we deduce that $P(k-\ell)$ is true and therefore that all of $P(k-\ell), P(k-\ell+1), \ldots, P(k-\ell+m)$ are true. Thus $R(k-\ell)$ is true and by induction, we conclude that $R(\ell)$ is true for all $\ell \geq 0$. Thus $P(n)$ is true for all $n \geq k$.

We've now shown that $P(n)$ holds for $n \geq k$ and $n \leq k$ so we may conclude it holds for all k. $\square$

Problem 2. Assume that n and k are non-negative integers. Find a formula for

$$\sum_{m=0}^{n} \binom{m}{k}$$

in terms of n and k and prove it by induction. (Hint: I suggest doing induction on n.)

Solution. We will prove that

$$\sum_{m=0}^{n} \binom{m}{k} = \binom{n+1}{k+1}$$

for $n \geq k$ by induction on n. When $n = 0$, we have $\sum_{m=0}^{n} \binom{m}{k} = \binom{0}{k}$, which is zero for $k > 0$ and is 1 for $k = 0$. On the other hand, $\binom{n+1}{k+1} = \binom{1}{k+1}$ is zero for $k > 0$ and is 1 for $k = 0$. This provides the base case for induction.

Now we consider the inductive step. Assuming the formula for n, we prove it for $n+1$. Adding $\binom{n+1}{k}$ to both sides of the equation, we get

$$\binom{n+1}{k} + \sum_{m=0}^{n} \binom{m}{k} = \binom{n+1}{k} + \binom{n+1}{k+1}.$$

The left side is the same as $\sum_{m=0}^{n+1} \binom{m}{k}$ and the right side is the same as $\binom{n+2}{k+2}$, by Pascal's identity. This proves the inductive step and completes the proof of the formula for all n. $\square$