Math 2001-002 Spring 2014
Homework 24

Last revised: March 10, 2014 at 10:01am

Problem 1. Prove that no non-empty set is disjoint from itself.

Solution. Suppose for the sake of contradiction that S is a non-empty set that is disjoint from itself. As S is non-empty, we may select some $x \in S$. Then $x \in S \cap S$, so that $S \cap S \neq \varnothing$. Therefore S is not disjoint from itself contrary to our assumption. □

Problem 2. How many ways are there to rearrange the list $(1, 2, 3, 4, 5)$ such that each of 1, 3, and 5 does not wind up in the same place? Justify your answer.

Solution. The answer is 64. Let S be the set of rearrangements of the list $(1, 2, 3, 4, 5)$. This contains $5! = 120$ elements. Let A_i be the subset of rearrangements that hold i in place. Then we want to count $|S - A_1 - A_3 - A_5|$. We can try to compute this number as

$$|S| - |A_1| - |A_3| - |A_5|$$

but then we will have subtracted everything in $A_1 \cap A_3$, $A_1 \cap A_5$ and $A_3 \cap A_5$ twice. We can try adding those back in:

$$|S| - |A_1| - |A_3| - |A_5| + |A_1 \cap A_3| + |A_1 \cap A_5| + |A_1 \cap A_3|$$

Now if $x \in A_1 \cap A_3$ and $x \notin A_5$ it will be counted

$$1 - 1 - 1 + 1 = 0$$

times, but if $x \in A_1 \cap A_3 \cap A_5$, it will be counted

$$1 - 1 - 1 - 1 + 1 + 1 + 1 = 1$$

times. To correct for this, we subtract off $|A_1 \cap A_3 \cap A_5|$ and get

$$|S| - |A_1| - |A_3| - |A_5| + |A_1 \cap A_3| + |A_1 \cap A_5| + |A_3 \cap A_5| - |A_1 \cap A_3 \cap A_5|.$$

To conclude, we should compute these values. The size of A_i is $4!$; the size of $A_i \cap A_j$ is $3!$ if $i \neq j$. And the size of $A_1 \cap A_3 \cap A_5$ is $2! = 2$. Therefore the number of rearrangemenets that move 1 and 3 and 5 is

$$5! - 3 \times 4! - 3 \times 3! + 2! = 64.$$

□

Problem 3. Prove that if k and n are integers such that $0 < k < n$ then k is not divisible by n.

Solution. Suppose, for the sake of contradiction, that k were divisible by n. Then we would have $k = mn$ for some integer m. As both k and n are positive, m must be positive as well. In particular, $m = 1 + q$ for some integer q with $q \geq 0$. Therefore $mn = (1 + q)n = n + qn \geq n + 0 = n$. On the other hand $mn = k < n$. This is absurd, so the premise could not have been possible. □

Problem 4. A number is called *rational* if it can be expressed as u/v where u and v are integers (and $v \neq 0$). Prove by contradiction that $\sqrt{2}$ is not rational. You may use the following fact about the number 2 that was proved in class: If x is an integer and $2|x^2$ then $2|x$. (Hint: Choose u to be the smallest positive integer such that there is a positive integer v with $u/v = \sqrt{2}$ and find a contradiction.)

Solution. Choose u and v as in the hint. If $\sqrt{2} = u/v$ then $u^2 = 2v^2$. Then $2|u^2$ so $2|u$. But this means $u = 2w$ for some positive integer w so that $2w^2 = v^2$. Then $2|v^2$ so $2|v$ so $v = 2x$ for some positive integer x. Therefore $w^2 = 2v^2$ so $w/v = \sqrt{2}$. But $w < u$ so this contradicts the assumption that u was the smallest positive integer such that $\sqrt{2} = u/v$ for some positive integer v. □