Math 2001-002 Spring 2014
Final Exam

Saturday, May 3, 2014

INSTRUCTIONS: Work alone, with no materials except pen, pencil, paper, and brain. Write your answers on the additional sheets provided. Make sure that every solution is numbered and that your full name appears on every page you turn in. Submit your answers with this cover sheet. You may keep the problem sheet.

Justification is required for all answers. Unless otherwise specified, solutions must be written in complete sentences. Answers will be graded on clarity in addition to correctness, so write neatly and express yourself clearly.

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Problem 1. (4 points) A *multiset* is a collection of objects in which order does not matter but repetition *is allowed*. The multiset containing the objects $a_1, \dots, a_n$ is denoted $\langle a_1, \dots, a_n \rangle$. Two multisets A and B are considerd to be the same if each object appears the same number of times in A as it does in B. How many *different* multisets are listed below? (Some of the entries below are the same multiset written in different ways.)

$$\begin{array}{ccc} \langle 1, \varnothing, 1 \rangle & \langle \varnothing, 1, \varnothing \rangle & \langle\, \rangle \\ \langle \varnothing, 1 \rangle & \langle 1, 1, \varnothing \rangle & \langle 1, \varnothing \rangle \end{array}$$

Problem 2. (8 points) Determine which of the following statements are true and which are false. Justify each answer in exactly one sentence.

(a) (2 points) $\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, x + y = 0$

(b) (2 points) $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, x + y = 0$

(c) (2 points) $\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, x + y = 0$

(d) (2 points) $\exists x \in \mathbb{N}, \exists y \in \mathbb{N}, x + y = 0$

Problem 3. (5 points) Suppose that $x_1, x_2, x_3, \dots$ is a sequence of real numbers. The following sentence means that this sequence converges to x:

$$\forall \epsilon \in \mathbb{R}, \big(\epsilon > 0 \implies \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, (n > N \implies |x_n - x| < \epsilon)\big)$$

Write a symbolic expression *without using* NOT *or* $\neg$ that means "The sequence does not converge to x." You may use $a \geq b$ for the negation of $a < b$.

Problem 4. (5 points) Prove or give a counterexample: For any sets A, B, and C,

$$A \times (B \times C) = (A \times B) \times C.$$

Problem 5. (6 points) Write a symbolic sentence that means "For any positive real number x there are exactly two real numbers y such that $y^2 = x$."

Problem 6. (8 points) Suppose that a, b, c, and d are integers and that $d|c$. Prove that if $a \equiv b \pmod{c}$ then $a \equiv b \pmod{d}$. (Recall that $x|y$ means that there is an integer z such that $xz = y$ and $x \equiv y \pmod{z}$ means that $z|x - y$.)

Problem 7. (8 points) Prove that for any natural number n,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Problem 8. (8 points) Suppose that $f : A \to B$ is a bijection and $g : B \to C$ and $h : B \to C$ are two functions such that $g \circ f = h \circ f$. Prove that $g = h$.

Problem 9. (12 points) For each $n \in \mathbb{N}$, let S_n be the set of all n-digit lists whose entries are 0s and 1s. Let T_n be the set of all n-digit lists of 0s and 1s that contain an even number of 1s. For example:

$$S_2 = \{(0,0), (0,1), (1,0), (1,1)\} \qquad T_2 = \{(0,0), (1,1)\}$$

(a) (2 points) Write down all elements of S_3 and T_3. Complete sentences are not required.

(b) (6 points) Suppose that $n \geq 1$ is a natural number. Consider the function $f : T_n \to S_{n-1}$ defined by

$$f\big((a_1, \dots, a_n)\big) = (a_1, \dots a_{n-1}).$$

(For example, $f\big((1,1,0,1,0,1)\big) = (1,1,0,1,0)$.) Show that f is a bijection.

(c) (4 points) Compute the size of T_n for all $n \in \mathbb{N}$ and prove your answer is correct.

Problem 10. (12 points) In this problem, we will prove that there are infinitely many prime numbers.

(i) (6 points) Suppose $p_1, \dots, p_k$ are prime numbers. Prove that $p_1 \cdots p_k + 1$ is not divisible by any of the p_i.

(ii) (6 points) Use the previous part to show that there are infinitely many prime numbers. (Hint: Use the fact, proved on homework, that every integer ≥ 2 is divisible by a prime number.)