Math 2001-002 Spring 2014
Midterm Exam 2

Thursday, March 20, 2014

INSTRUCTIONS: Work alone, with no materials except pen, pencil, paper, and brain. Write your answers on the additional sheets provided. Make sure that every solution is numbered and that your full name appears on every page you turn in. Submit your answers with this cover sheet. You may keep the problem sheet.

Justification is required for all answers. Unless otherwise specified, solutions must be written in complete sentences. Answers will be graded on clarity in addition to correctness, so write neatly and express yourself clearly.

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Problem 1. (3 points) Which of the following is not logically equivalent to the others?

A) $X \implies Y$

B) $\neg X \vee Y$

C) $\neg Y \implies \neg X$

D) $Y \implies X$

Solution. We can verify that $X \implies Y$ is equivalent to $\neg X \vee Y$ with a truth table:

X	Y	$\neg X$	$\neg X \vee Y$	$X \implies Y$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

This also implies that $\neg Y \implies \neg X$ is equivalent to $\neg\neg Y \vee \neg X$, which is equivalent to $Y \vee \neg X$, which is equivalent to $\neg X \vee Y$.

On the other hand, if $X = T$ and $Y = F$ then $X \implies Y$ is false while $Y \implies X$ is true. Therefore $Y \implies X$ is not equivalent to any of the others. $\square$

Problem 2. (5 points) Prove or disprove: If S is a set with n elements and R is an equivalence relation on S with m equivalence classes then n is divisible by m.

Solution. Let $S = \{1, 2, 3\}$ and let $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$. Then the equivalece classes of R are $\{1, 2\}$ and $\{3\}$. The number of equivalence classes is 2, which does not divide $|S| = 3$. $\square$

Problem 3. (9 points) Let Q and R be relations. Define:

$$Q \circ R = \{(x, z) \ : \ \exists y, (y, z) \in Q \text{ and } (x, y) \in R\}$$

For parts (i), (ii), and (iii), assume that Q and R are defined as follows:

$$Q = \{(2, 3), (3, 2), (5, 3), (6, 7)\}$$
$$R = \{(1, 3), (1, 5), (1, 6), (2, 2), (2, 6), (3, 6)\}$$

For part (iv), the relation R is arbitrary.

(i) (1 point) Draw pictures illustrating Q and R as arrows. Complete sentences are not required.

(ii) (1 point) Compute $Q \circ R$. Complete sentences are not required.

Solution.

$$Q \circ R = \{(1, 2), (1, 3), (1, 7), (2, 3), (2, 7), (3, 7)\}$$

$\square$

(iii) (1 point) Draw a picture llustrating $Q \circ R$ as arrows. Complete sentences are not required.

(iv) (6 points) Prove that a relation R is transitive if and only if $R \circ R \subset R$.

Solution. Suppose first that R is transitive. We must show that, for any $(x, z) \in R \circ R$ we also have $(x, z) \in R$. By definition, if $(x, z) \in R$ then there is some y such that $(x, y) \in R$ and $(y, z) \in R$. But because R is transitive, this implies that $(x, z) \in R$, as desired.

Now suppose that $R \circ R \subset R$. To show that R is transitive, we must verify that, whenever $(x, y) \in R$ and $(y, z) \in R$ we also have $(x, z) \in R$. To see this, observe that if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R \circ R$. As $R \circ R \subset R$, this implies that $(x, z) \in R$, as desired. $\square$

Problem 4. (7 points) On this problem, your answers may use integers, the symbol n, and the operations of addition, subtraction, multiplication, and division. In particular, you may not use binomial coefficients, factorials, or ellipses.

(i) (3 points) Find the coefficient of x^3y^{n-3} in $(x+y)^n$. Complete sentences are not required.

Solution.

$$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6}$$

$\square$

(ii) (4 points) Find the coefficient of $x^3y^4z^{n-7}$ in $(x+y+z)^n$.

Solution. Let $w = y+z$. The coefficient of x^3w^{n-3} in $(x+w)^n$ is $\binom{n}{3}$, as above. That is, we could write

$$(x+y+z)^n = \cdots + \binom{n}{3}x^3(y+z)^{n-3} + \cdots$$

The coefficient of y^4z^{n-7} in $(y+z)^{n-3}$ is $\binom{n-3}{4}$, so we get

$$(x+y+z)^n = \cdots + \binom{n}{3}x^3\left(\cdots + \binom{n-3}{4}y^4z^{n-7} + \cdots\right) + \cdots$$

After distributing, we get

$$(x+y+z)^n = \cdots + \binom{n}{3}\binom{n-3}{4}x^3y^4z^{n-7} + \cdots$$

Therefore the coefficient is

$$\binom{n}{3}\binom{n-3}{4} = \frac{n!(n-3)!}{3!\,(n-3)!\,4!\,(n-7)!} = \frac{n!}{6(24)(n-7)!} = \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{144}.$$

$\square$

Solution. Here is a second solution, by grouping x and y together:

$$\begin{aligned}
(x+y+z)^n &= ((x+y)+z)^n \\
&= \cdots + \binom{n}{7}(x+y)^7z^{n-7} + \cdots \\
&= \cdots + \binom{n}{7}(\cdots + \binom{7}{3}x^3y^4 + \cdots)z^{n-7} + \cdots \\
&= \cdots + \binom{n}{7}\binom{7}{3}x^3y^4z^{n-7} + \cdots
\end{aligned}$$

Therefore the coefficient is

$$\begin{aligned}
\binom{n}{7}\binom{7}{3} &= \frac{n!7!}{7!(n-7)!3!4!} \\
&= \frac{n!}{(n-7)!3!4!} \\
&= \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{144}
\end{aligned}$$

$\square$

Problem 5. (6 points) Prove that, for any real numbers a, b, and c,

$ab = ac$ if and only if $a = 0$ or $b = c$.

Solution. Suppose for the sake of contradiction that $a \neq 0$ and $b \neq c$. Then we may divide by a in the equation $ab = ac$ to obtain $b = c$. This contradicts the assumption that $b \neq c$, so the original assumption must have been false. That is, we must have $a = 0$ or $b = c$. $\square$

Solution. The assertion is equivalent to the assertion that

If $ab = ac$ and $a \neq 0$ then $b = c$.

Assume that $ab = ac$ and $a \neq 0$. Then we may divide by a since $a \neq 0$ to obtain $b = c$. $\square$

Problem 6. (4 points) Consider an infinite parade of elephants, where there is one elephant for each integer. Assume that the n-th elephant is pink if and only if the $(n + 2)$-th elephant is pink. Describe the minimal amount of additional information you would need to conclude that every elephant in the parade is pink. Your answer to this problem should be justified, but it does not have to be a fully rigorous proof.

Solution. At least one of the odd-numbered elephants is pink and at least one of the even-numbered elephants is pink.

Suppose that the k-th elephant is pink. Then by induction, the $(k + 2m)$-th elephant will be pink, for every $m \geq 0$. Likewise, the $(k - 2m)$-th elephant will be pink for every $m \geq 0$. Thus, if an even-numbered elephant is pink then *all* of the even-numbered elephants are pink. Likewise, if an odd-numbered elephant is pink then all of the odd-numbered elephants are pink. Since every integer number is either even or odd, if at least one even-numbered is pink and at least one odd-numbered elephant is pink then all elephants in the parade are pink. $\square$

Problem 7. (6 points) Call a collection of sets $A_1, \ldots, A_n$ *triple-wise disjoint* if

$$A_i \cap A_j \cap A_k = \varnothing$$

whenever i, j, and k are pairwise distinct[1] indices between 1 and n. Prove that if $A_1, \ldots, A_n$ are triple-wise disjoint then the following formula holds:

$$|A_1 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| \quad - \sum_{1 \leq i < j \leq n} |A_i \cap A_j|$$

You may *not* use the inclusion-exclusion formula on this problem.

Solution. For each $x \in A_1 \cup \cdots \cup A_n$, we evaluate how many times it is counted on the left side of the equation and how many times it is counted on the right side of the equation. We will see that these numbers are the same, so the two sides must be equal.

If $x \in A_1 \cup \cdots \cup A_n$ then x could be contained in exactly one of the A_i or it could be contained in two of them. The condition $A_i \cap A_j \cap A_k = \varnothing$ (for pairwise distinct i,j, k) implies that it cannot be contained in more than two of the A_i-s.

If x is contained in exactly one of the A_i-s then x contributes 1 to the sum $\sum_{i=1}^{n} |A_i|$ and 0 to the sum $\sum_{1 \leq i < j \leq n} |A_i \cap A_j|$. If x is contained in two of the A_i-s, then x contributes 2 to $\sum_{i=1}^{n} |A_i|$ but it also contributes 1 to the sum $\sum_{1 \leq i < j \leq n} |A_i \cap A_j|$. Either way, x contributes a total of 1 to the difference

$$\sum_{i=1}^{n} |A_i| \quad - \sum_{1 \leq i < j \leq n} |A_i \cap A_j|.$$

Since x also contributes exactly 1 to $|A_1 \cup \cdots \cup A_n|$, we may conclude that

$$|A_1 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| \quad - \sum_{1 \leq i < j \leq n} |A_i \cap A_j|.$$

$\square$

[1] For i, j, and k to be pairwise distinct means that $i \neq j$ and $i \neq k$ and $j \neq k$.

Solution. Here is a solution by induction. If $n = 0$ then both sides of the equation are 0. If $n = 1$ then the left side is $|A_1|$ and the right side is $|A_1|$. If $n = 2$ then the left side is $|A_1 \cup A_2|$ and the right side is $|A_1| + |A_2| - |A_1 \cap A_2|$. This formula was proved in the textbook and in class.

Now we assume for the sake of induction that

$$|A_1 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| \quad - \sum_{1 \le i < j \le n} |A_i \cap A_j|$$

and demonstrate that

$$|A_1 \cup \cdots \cup A_{n+1}| = \sum_{i=1}^{n+1} |A_i| \quad - \sum_{1 \le i < j \le n} |A_i \cap A_j|. \tag{$*$}$$

Let $B = A_1 \cup \cdots \cup A_n$. Then the left side of this equation can be written $|B \cup A_{n+1}|$, and we know that

$$\begin{aligned}
|B \cup A_{n+1}| &= |B| + |A_{n+1}| - |B \cap A_{n+1}| \\
&= \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + |A_{n+1}| - |B \cap A_{n+1}| \\
&= \sum_{i=1}^{n+1} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| - |B \cap A_{n+1}|
\end{aligned}$$

To prove Equation $(*)$ we need to show that

$$|B \cap A_{n+1}| = \sum_{i=1}^{n} |A_i \cap A_{n+1}|.$$

We will prove this by induction on n. It will be easier to state this as a lemma with the sets in a different order:

Lemma. If $B_1, \ldots, B_n$ are sets that are triple-wise disjoint then

$$|B_1 \cap (B_2 \cup \cdots \cup B_n)| = \sum_{i=2}^{n} |B_1 \cap B_i|.$$

We prove this by induction on n. If $n = 0$ then both sides are 0; if $n = 1$ then both sides are $|A_1 \cap A_2|$. Assuming for the sake of induction that

$$|B_1 \cap (B_2 \cup \cdots \cup B_n)| = \sum_{i=2}^{n} |B_1 \cap B_i|$$

we prove that

$$|B_1 \cap (B_2 \cup \cdots \cup B_{n+1})| = \sum_{i=2}^{n+1} |B_1 \cap B_i|.$$

Let $C = B_2 \cup \cdots \cup B_n$. Then we have

$$\begin{aligned}
|B_1 \cap (C \cup B_{n+1})| &= |(B_1 \cap C) \cup (B_1 \cap B_{n+1})| \\
&= |B_1 \cap C| + |B_1 \cap B_{n+1}| - |B_1 \cap C \cap B_{n+1}| \\
&= \sum_{i=2}^{n} |B_1 \cap B_i| + |B_1 \cap B_{n+1}| - |B_1 \cap C \cap B_{n+1}| \\
&= \sum_{i=2}^{n+1} |B_1 \cap B_{n+1}| - |B_1 \cap C \cap B_{n+1}|
\end{aligned}$$

But any element of $B_1 \cap C \cap B_{n+1}$ is contained in $B_1 \cap C \cap B_{n+1}$ for some $1 < i < n+1$. Sicne the B_i are triple-wise disjoint, this means that $B_1 \cap C \cap B_{n+1} = \varnothing$, so we conclude

$$\begin{aligned} |B_1 \cap (B_1 \cup \cdots \cup B_{n+1})| &= |B_1 \cap (C \cup B_{n+1})| \\ &= \sum_{i=2}^{n+1} |B_1 \cap B_{n+1}|. \end{aligned}$$

This completes the proof of the lemma. It also completes the solution to the problem. $\square$