Problem 1. (32 points) Short answers.

(a) (3 points) Call an integer n *threeven* if it is divisible by 3 and *throdd* if $n = 3k + 1$ for some integer k. Disprove the following statement: Every integer is either threeven or throdd.

Solution. 2 is not threeven since it is not a multiple of 3. It is not throdd, because if it were then there would be an integer such that $2 = 3k + 1$. But then $1 = 3k$ so 1 would be divisible by 3 and we know this is not the case. $\square$

(b) (2 points) Suppose that n is a natural number and that S and T are sets. If $|S| = n$ and $|T| = n$ and $S \subset T$ then $S = T$. (True / False)

Solution. True $\square$

(c) (2 points) Which of the following are logical opposites of $X \implies Y$?

(A) $Y \implies X$ (B) $\text{NOT } Y \implies \text{NOT } X$ (C) $X \wedge \text{NOT } Y$ (D) $\text{NOT } X \vee Y$

(Scoring: 1/2 point for each correct answer circled, 1/2 point for each incorrect answer not circled.)

Solution. (C) $\square$

(d) (3 points) Find a counterexample to the assertion, "Every integer larger than 4 is a sum of two primes". Justify your answer, briefly.

Solution. 27 cannot be a sum of two primes. If it were then one of the primes would have to be 2 since the sum of two odd numbers is always even and all primes other than 2 are odd. But then the only way this could happen is if $27 = 25 + 2$ and 25 is not prime. $\square$

(e) (2 points) The empty set is an element of every set. (True / False)

Solution. False $\square$

(f) (3 points) Write down the elements of the power set of $\{\varnothing, \{1\}\}$.

Solution. $\varnothing, \{\varnothing\}, \{\{1\}\}, \{\varnothing, \{1\}\}$ $\square$

(g) (4 points) Let S be a set with n elements, where $n \geq 2$. How many 3-element lists without repetition are there whose elements are subsets of S? Express your answer using addition, subtraction, multiplication, division, exponentiation, the integers, and the symbol n. Justify your answer, briefly.

Solution. There are 2^n subsets of S. The number of 3-element lists drawn from a set with 2^n elements is $(2^n)_3 = 2^n(2^n - 1)(2^n - 2)$. $\square$

(h) (3 points) Is it possible for a set to be disjoint from itself? Either give an example or explain why no example exists.

Solution. Yes: $\varnothing \cap \varnothing = \varnothing$ so $\varnothing$ is disjoint from itself. $\square$

(i) (2 points) Suppose that S is a finite set. Put the following three numbers in order of size using the symbols $<$ and $\leq$:

(A) The number of subsets of S of size k.

(B) The number of lists of length k, without repetition, drawn from S.

(C) The number of lists of length k, with repetition allowed, drawn from S.

Solution. (A) $\leq$ (B) $\leq$ (C). Indeed, a list with repetition disallowed is in particular a list with repetition allowed, so there are at least as many lists with repetition allowed as there are with it disallowed. On the other hand, if we can turn a subset into a list with repetition disallowed by choosing an order for the elements of the subset. Therefore there are at least as many sets with repetition disallowed as there are subsets. Finally, if $k = 0$ then all three of these numbers are 1, so that none of the inequalities is strict. □

(j) (2 points) What is the truth value of the following sentence?

$$X_1 \wedge (X_1 \to X_2) \wedge (X_2 \to X_3) \wedge (X_3 \to X_4) \wedge (X_4 \to X_5) \wedge (X_5 \to X_6) \wedge \text{NOT } X_6$$

(A) True

(B) False

(C) It depends on the parameters

Solution. The sentence is false. Suppose there were a choice of truth values of the X_i that made the sentence true. Then X_1 would be true and X_6 would be false. But $X_1 \to X_2$ would have to be true, which means that X_2 be true since X_1 is already known to be true. But then $X_2 \to X_3$ would also be true, so that X_3 would be true. Continuing to reason this way, we discover that X_6 would be true. On the other hand, we already saw that it had to be false. Therefore there is no way for the sentence to be true. No matter what the truth values of the X_i, the sentence will always be false. □

(k) (3 points) Let a and b be integers with $b > a$. How many subsets are there of $\mathbb{Z}$ whose smallest element is a and whose largest element is b?

Solution. To give such a set amounts to the same as giving a subset of the set $\{a+1, \ldots, b-1\}$, which has $b - a - 1$ elements. There are 2^{b-a-1} such subsets. □

(l) (3 points) In calculus, a function f is called *continuous* if, for every x and every positive number ϵ, there is a positive number δ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. This can be written in symbols like this:

$$\forall x, \forall \epsilon > 0, \exists \delta > 0, \forall y, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Write a symbolic expression meaning "f is not continuous" without using NOT. You may use $a \geq b$ as the logical opposite of $a < b$.

Solution. $\exists x, \exists \epsilon > 0, \forall \delta > 0, \exists y, |x - y| < \delta \wedge |f(x) - f(y)| \geq \epsilon$ □

Problem 2. (6 points) Prove that an integer x divides $x+6$ if and only if x divides 6.

Solution. Suppose that x divides $x+6$. Then there is an integer y such that $x+6=yx$. But then $6=yx-x=(y-1)x$. As $y-1$ is an integer, this means $x|6$.

Conversely, suppose that x divides 6. Then there is an integer y such that $6=yx$. But then $6+x=yx+x=(y+1)x$. As $y+1$ is an integer, this means that x divides $6+x$. □

Problem 3. (6 points) The logical operator NAND is defined by

$$X \text{ NAND } Y = \text{NOT } (X \wedge Y).$$

(a) (2 points) Write down the truth table for $X \text{ NAND } Y$.

Solution. Here is the table:

X	Y	$X \wedge Y$	$X \text{ NAND } Y$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

□

(b) (4 points) Write each of the following expressions as a formula involving only NAND and variables. You do not need to use complete sentences.

(i) (2 points) NOT X

Solution. $X \text{ NAND } X$ □

(ii) (2 points) $X \wedge Y$

Solution. $X \wedge Y = \text{NOT } (X \text{ NAND } Y) = (X \text{ NAND } Y) \text{ NAND } (X \text{ NAND } Y)$ □

Problem 4. (10 points) In this problem we will prove the distributive property of multiplication over addition. Contrary to standard practice in this class, *you are not allowed use standard facts from arithmetic in this problem.* However, you may use the following facts that were proved in the textbook and on the homework:

(I) If X and Y are sets then $|X \times Y| = |X| \times |Y|$.

(II) If X and Y are sets then $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$.

(III) If X and Y are sets then $X \times (Y \cap Z) = (X \times Y) \cap (X \times Z)$.

(IV) If X and Y are sets then $|X \cup Y| = |X| + |Y| - |X \cap Y|$.

Solve the following problems:

(a) (3 points) Express the size of $X \times (Y \cup Z)$ in terms of the sizes of X, Y, Z, and $Y \cap Z$. Justify your answer.

Solution. We may compute:

$$\begin{aligned} |X \times (Y \cup Z)| &= |X| \times |Y \cup Z| && \text{By (I).} \\ &= |X| \times (|Y| + |Z| - |Y \cap Z|) && \text{By (IV).} \end{aligned}$$

□

(b) (3 points) Express the size of $(X \times Y) \cup (X \times Z)$ in terms of the sizes of X, Y, Z, and $|Y \cap Z|$. Justify your answer.

Solution. We can compute:

$$\begin{aligned}
|(X \times Y) \cup (X \times Z)| &= |X \times Y| + |X \times Z| - |(X \times Y) \cap (X \times Z)| && \text{By (IV).} \\
&= |X \times Y| + |X \times Z| - |X \times (Y \cap Z)| && \text{By (III)} \\
&= |X| \times |Y| + |X| \times |Z| - |X| \times |Y \cap Z| && \text{By (I).}
\end{aligned}$$

□

(c) (4 points) Using the previous parts of this problem, prove the distributive formula, $a(b+c) = ab + ac$ for non-negative integers a, b, and c.

Solution. Select sets X, Y, and Z with $|X| = a$, $|Y| = b$, $|Z| = c$ and $Y \cap Z = \varnothing$. For example, we could take

$$\begin{aligned}
X &= \{1, \ldots, a\} \\
Y &= \{1, \ldots, b\} \\
Z &= \{b+1, \ldots, b+c\}.
\end{aligned}$$

Then

$$\begin{aligned}
a(b+c) &= |X| \times (|Y| + |Z|) && \text{Substitution} \\
&= |X| \times (|Y| + |Z| - |Y \cap Z|) && \text{Since } Y \cap Z = \varnothing \\
&= |X \times (Y \cup Z)| && \text{By Part (a)} \\
&= |(X \times Y) \cup (X \times Z)| && \text{By Property (II)} \\
&= |X| \times |Y| + |X| \times |Z| - |X| \times |Y \cap Z| && \text{By Part (b)} \\
&= |X| \times |Y| + |X| \times |Z| && \text{Since } Y \cap Z = \varnothing \\
&= ab + ac && \text{Substitution.}
\end{aligned}$$

□

Problem 5. (10 points) Let S be a set. A *partition* of S into three subsets is a list (T, U, V) where T, U, and V are all subsets of S such that

(i) $T \cup U \cup V = S$ and

(ii) T, U, and V are pairwise disjoint.

To help you understand the definition, here are the three ways to partition the set $\{1\}$ into 3 subsets:

$$(\{1\}, \varnothing, \varnothing), (\varnothing, \{1\}, \varnothing), \text{ and } (\varnothing, \varnothing, \{1\}).$$

(a) (1 point) Let $S = \varnothing$. List all partitions of S into 3 subsets.

Solution. There is just one: $(\varnothing, \varnothing, \varnothing)$. □

(b) (1 point) Let $S = \{1, 2\}$. List all partitions of S into 3 subsets.

Solution. There are 9:

$$\begin{array}{ccc}
(\varnothing, \varnothing, \{1,2\}) & (\varnothing, \{1\}, \{2\}) & (\varnothing, \{2\}, \{1\}) \\
(\varnothing, \{1,2\}, \varnothing) & (\{1\}, \varnothing, \{2\}) & (\{1\}, \{2\}, \varnothing) \\
(\{2\}, \varnothing, \{1\}) & (\{2\}, \{1\}, \varnothing) & (\{1,2\}, \varnothing, \varnothing).
\end{array}$$

□

(c) (1 point) Conjecture a formula for the number of partitions of a set with n elements into 3 subsets. Express your answer using addition, subtraction, multiplication, division, exponentiation, the integers, and the symbol n.

Solution. The number of ways of partitioning a set with n elements into 3 subsets is 3^n. □

(d) (4 points) Prove that your formula is correct.

Solution. We could describe a partition of S into three subsets by going through the elements of S one by one, and for each one asking "Which of the three subsets is this element in?" For each of the n elements of S we therefore have 3 choices, which can be arbitrary. There are 3 possibilities for the first element; for each of those, there are 3 possibilities for the second element; once the answers for the first and second elements have been chosen, there are 3 possibilities for the third, and so on. In total, there are

$$3 \times 3 \times \cdots \times 3$$

possibilities, where we have multiplied one 3 for each element of S. There are n such elements, so the product is 3^n. □

(e) (3 points) Make a reasonable definition of a partition of a set S into k subsets and conjecture a formula for the number of ways to partition a set with n elements into k subsets.

Solution. A partition of S into k subsets is a list

$$(T_1, \ldots, T_k)$$

in which each T_i is a subset of S, and

(i) $\bigcup_{i=1}^{k} T_i = S$, and

(ii) the T_i are pairwise disjoint.

If S has n elements then the number of partitions of S into k subsets is n^k. □

Problem 6. (4 points) Imagine a conversation with someone who understands sets but has never heard of a number. Explain as much as you can to him about numbers using your common knowledge of set theory.