**Problem 1.** (32 points) Short answers.

(a) (3 points) Call an integer $n$ *threeven* if it is divisible by 3 and *throdd* if $n = 3k + 1$ for some integer $k$. Disprove the following statement: Every integer is either threeven or throdd.

(b) (2 points) Suppose that $n$ is a natural number and that $S$ and $T$ are sets. If $|S| = n$ and $|T| = n$ and $S \subset T$ then $S = T$. (True / False)

(c) (2 points) Which of the following are logical opposites of $X \implies Y$?

(A) $Y \implies X$ (B) $\text{NOT}\, Y \implies \text{NOT}\, X$ (C) $X \wedge \text{NOT}\, Y$ (D) $\text{NOT}\, X \vee Y$

(Scoring: 1/2 point for each correct answer circled, 1/2 point for each incorrect answer not circled.)

(d) (3 points) Find a counterexample to the assertion, "Every integer larger than 4 is a sum of two primes". Justify your answer, briefly.

(e) (2 points) The empty set is an element of every set. (True / False)

(f) (3 points) Write down the elements of the power set of $\{\varnothing, \{1\}\}$.

(g) (4 points) Let $S$ be a set with $n$ elements, where $n \geq 2$. How many 3-element lists without repetition are there whose elements are subsets of $S$? Express your answer using addition, subtraction, multiplication, division, exponentiation, the integers, and the symbol $n$. Justify your answer, briefly.

(h) (3 points) Is it possible for a set to be disjoint from itself? Either give an example or explain why no example exists.

(i) (2 points) Suppose that $S$ is a finite set. Put the following three numbers in order of size using the symbols $<$ and $\leq$:

(A) The number of subsets of $S$ of size $k$.

(B) The number of lists of length $k$, without repetition, drawn from $S$.

(C) The number of lists of length $k$, with repetition allowed, drawn from $S$.

(j) (2 points) What is the truth value of the following sentence?

$$X_1 \wedge (X_1 \to X_2) \wedge (X_2 \to X_3) \wedge (X_3 \to X_4) \wedge (X_4 \to X_5) \wedge (X_5 \to X_6) \wedge \text{NOT}\, X_6$$

(A) True

(B) False

(C) It depends on the parameters

(k) (3 points) Let $a$ and $b$ be integers with $b > a$. How many subsets are there of $\mathbb{Z}$ whose smallest element is $a$ and whose largest element is $b$?

(l) (3 points) In calculus, a function $f$ is called *continuous* if, for every $x$ and every positive number $\epsilon$, there is a positive number $\delta$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. This can be written in symbols like this:

$$\forall x, \forall \epsilon > 0, \exists \delta > 0, \forall y, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Write a symbolic expression meaning "$f$ is not continuous" without using NOT. You may use $a \geq b$ as the logical opposite of $a < b$.

**Problem 2.** (6 points) Prove that an integer $x$ divides $x+6$ if and only if $x$ divides 6.

**Problem 3.** (6 points) The logical operator NAND is defined by

$$X \text{ NAND } Y = \text{NOT } (X \wedge Y).$$

(a) (2 points) Write down the truth table for $X$ NAND $Y$.

(b) (4 points) Write each of the following expressions as a formula involving only NAND and variables. You do not need to use complete sentences.

  (i) (2 points) NOT $X$

  (ii) (2 points) $X \wedge Y$

**Problem 4.** (10 points) In this problem we will prove the distributive property of multiplication over addition. Contrary to standard practice in this class, *you are not allowed use standard facts from arithmetic in this problem.* However, you may use the following facts that were proved in the textbook and on the homework:

(I) If $X$ and $Y$ are sets then $|X \times Y| = |X| \times |Y|$.

(II) If $X$ and $Y$ are sets then $X \times (Y \cap Z) = (X \times Y) \cap (X \times Z)$.

(III) If $X$ and $Y$ are sets then $|X \cup Y| = |X| + |Y| - |X \cap Y|$.

Solve the following problems:

(a) (3 points) Express the size of $X \times (Y \cup Z)$ in terms of the sizes of $X$, $Y$, $Z$, and $Y \cap Z$. Justify your answer.

(b) (3 points) Find an expression for the size of $(X \times Y) \cup (X \times Z)$ in terms of the sizes of $X$, $Y$, $Z$, and $Y \cap Z$ that is *different from the one you found in the previous part.* Justify your answer.

(c) (4 points) Using the previous parts of this problem, prove the distributive formula, $a(b+c) = ab + ac$ for non-negative integers $a$, $b$, and $c$. You may use the following additional property:

  (IV) If $X$ and $Y$ are sets then $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$.

**Problem 5.** (10 points) Let $S$ be a set. A *partition* of $S$ into three subsets is a list $(T, U, V)$ where $T$, $U$, and $V$ are all subsets of $S$ such that

(i) $T \cup U \cup V = S$ and

(ii) $T$, $U$, and $V$ are pairwise disjoint.

(a) (2 points) Explain why each of the following is a partition of $\{1\}$ into three subsets:

$$(\{1\}, \varnothing, \varnothing), (\varnothing, \{1\}, \varnothing), \text{ and } (\varnothing, \varnothing, \{1\}).$$

(b) (2 points) Explain why $(1, 2, \varnothing)$ is not a partition of $\{1, 2\}$ into three subsets.

(c) (1 point) Let $S = \varnothing$. List all partitions of $S$ into 3 subsets.

(d) (1 point) Let $S = \{1, 2\}$. List all partitions of $S$ into 3 subsets.

(e) (1 point) Conjecture a formula for the number of partitions of a set with $n$ elements into 3 subsets. Express your answer using addition, subtraction, multiplication, division, exponentiation, the integers, and the symbol $n$.

(f) (4 points) Prove that your formula is correct.

(g) (3 points) Make a reasonable definition of a partition of a set $S$ into $k$ subsets and conjecture a formula for the number of ways to partition a set with $n$ elements into $k$ subsets.

**Problem 6.** (4 points) Imagine a conversation with someone who understands sets but has never heard of a number. Explain as much as you can to him about numbers using your common knowledge of set theory.