Exploration 23

Math 2001–002, Fall 2016

November 28, 2016

Axiom 1. All integers are either ≥ 0 or < 0 and no integer is both ≥ 0 and < 0.

Theorem 2. *All natural numbers are either even or odd, and no natural number is both even and odd.*

Definition 3. Two sets A and B *have the same cardinality* if there is a bijection from A to B.

Theorem 4. *If $f : A \to B$ is a function then f is a bijection if and only if f has a two-sided inverse.*

Definition 5. If $f : A \to B$ is a function, a *left inverse* of f is a function $g : B \to A$ such that $g(f(a)) = a$ for all $a \in A$. A *right inverse* of f is a function $g : B \to A$ such that $f(g(b)) = b$ for all $b \in B$. A *two-sided inverse* of f is a function $g : B \to A$ that is both a left inverse of f and a right inverse of f.

Question 6. Construct a rigorous proof of the following theorem:

> **Theorem.** The sets $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.

You may want to define functions $f : \mathbb{N} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{N}$ by the following formulas:

$$
\begin{aligned}
f(n) &= -\frac{n}{2} \quad n \in \mathbb{N} \text{ and } n \text{ is even}, & \qquad g(z) &= -2z \quad \text{if } z \in \mathbb{Z} \text{ and } z < 0, \\
f(n) &= \frac{n-1}{2} \quad n \in \mathbb{N} \text{ and } n \text{ is odd}, & \qquad g(z) &= 2z+1 \quad \text{if } z \in \mathbb{Z} \text{ and } z \geq 0.
\end{aligned}
$$

If you use these functions, make sure to verify that they are well-defined!