# Exploration 19

Math 2001–002, Fall 2016

November 4, 2016

**Theorem.** *If $S$ is a nonempty subset of $\mathbb{N}$ then $S$ contains a smallest element.*

**Question 1.** Is it true that every subset of $\mathbb{Z}$ has a smallest element? What about every subset of $\mathbb{R}$?

1 *Proof of the theorem.* We will prove the following equivalent statement: If $S$ is a subset of $\mathbb{N}$ with no smallest element then $S = \varnothing$.

2 Suppose that $S$ is a subset of $\mathbb{N}$ with no smallest element.

3 We wish to show that that every $n \in \mathbb{N}$ is not an element of $S$.

4 We will use strong induction on $n$ to prove this.

5 Assume that $n \in \mathbb{N}$ and that, for all natural numbers $m$ in the range $1 \leq m < n$, we already know $m \notin S$.

6 We will prove that $n \notin S$.

7 Indeed, if $n$ were in $S$ then $n$ would be the smallest element of $S$.

8 We know that $S$ has no smallest element, so this means $n$ can't possibly be in $S$.

9 That is what we wanted to prove, so this completes the inductive step.

10 The induction completes the proof. Q.E.D.