Exploration 14

Math 2001–002, Fall 2016

October 17, 2016

Theorem 1. *Every positive integer is both even and odd.*

Proof. All positive integers are generated by repeatedly adding 1, so what we need to show is

> **Claim.** If x is a positive integer that is both even and odd then $x+1$ is both even and odd.

To prove this claim, suppose that x is a positive intger that is both even and odd. Since x is even and 1 is odd, $x+1$ is the sum of an even number and an odd number, so $x+1$ is odd. Since x is odd and 1 is odd, $x+1$ is the sum of an odd number and another odd number, so $x+1$ is even. Therefore $x+1$ is both even and odd, as required. □

Theorem 2. *All real numbers are equal.*

Proof. What we will show is the following claim:

> **Claim.** For all lists $x_1, \dots, x_n$ of n real numbers, $x_1 = x_2 = \cdots = x_n$.

We will prove this by induction on the length of the list, n, which is a positive integer. As in any induction proof, we have two things to show:

S1 For any list of 1 real number, all numbers in the list are equal.

S2 If for all lists of n real numbers $x_1, \dots, x_n$ we know that $x_1 = x_2 = \cdots = x_n$ then for all lists of $n+1$ real numbers, $y_1, y_2, \dots, y_{n+1}$ we have $y_1 = y_2 = \cdots = y_{n+1}$.

We know that **S1** is true because any real number equals itself. Let's prove **S2**. Assume that for all lists of n real numbers $x_1, \dots, x_n$ we have $x_1 = x_2 = \cdots x_n$. Suppose that $y_1, \dots, y_{n+1}$ is a list of $n+1$ real numbers. Then $y_1, \dots, y_n$ is a list of n real numbers, so by the inductive hypothesis, $y_1 = y_2 = \cdots = y_n$. On the other hand, $y_2, \dots, y_{n+1}$ is also a list of n real numbers, so $y_2 = y_3 = \cdots = y_{n+1}$. Putting these two facts together gives us

$$y_1 = y_2 = \cdots = y_n = y_{n+1}.$$

Therefore all of the real numbers in the list $y_1, \dots, y_{n+1}$ are the same, as required. This proves **S2** and therefore proves the claim. □

For the next theorem, we make use of a recursively defined sequence:

$$\begin{aligned} A_0 &= 2 \\ A_1 &= 3 \\ A_n &= 5B_{n-1} - 6B_{n-2} \end{aligned}$$

Theorem 3. *For all integers $n \geq 0$ we have $A_n = 2^n + 3^n$.*

Proof. We prove this by induction on n. We have two things to prove:

T1 $A_0 = 2^0 + 3^0$.

T2 If $A_n = 2^n + 3^n$ then $A_{n+1} = 2^{n+1} + 3^{n+1}$.

The proof of **T1** is not difficult: A_0 was defined to be 2 and $2^0+3^0 = 1+1 = 2$. Therefore $A_0 = 2^0 + 3^0$, as required.

Let's prove **T2**. Suppose that $A_n = 2^n + 3^n$. We know by the definition of A_{n+1} that

$$A_{n+1} = 5A_n - 6A_{n-1}.$$

We substitute in $A_n = 2^n + 3^n$ and $A_{n-1} = 2^{n-1} + 3^{n-1}$ and get

$$\begin{aligned} A_{n+1} &= 5(2^n + 3^n) - 6(2^{n-1} + 3^{n-1}) \\ &= (2+3)(2^n + 3^n) - (2 \times 3)(2^{n-1} + 3^{n-1}) \\ &= 2^{n+1} + 2 \times 3^n + 3 \times 2^n + 3^{n+1} - 3 \times 2^n - 2 \times 3^n \\ &= 2^{n+1} + 3^{n+1}, \end{aligned}$$

which is exactly what we needed to show. This completes the proof of **T2** and also the proof of the theorem. □

For the next theorem, we make use of a recursively defined sequence:

$$\begin{aligned} B_0 &= 2 \\ B_1 &= 5 \\ B_n &= 5B_{n-1} - 6B_{n-2} \end{aligned}$$

Theorem 4. *For all integers $n \geq 0$ we have $B_n = 2^n + 3^n$.*