Exploration 10

Math 2001–002, Fall 2016

September 28, 2016

Axiom 1. The positive integers are generated by the following rules:

N1 1 is a positive integer.

N2 If x is a positive integer then $x + 1$ is a positive integer.

Axiom 2. The integers are generated by the following rules:

Z1 0 is an integer.

Z2 If x is an integer then $x + 1$ is an integer.

Z3 If x is an integer then $x - 1$ is an integer.

Question 3. For each of the theorem statements on the next page, go through the following steps:

1) Diagram the sentence in question.

2) Create at least 3 examples to test whether the statement is true.

3) Phrase the sentence in a form suitable for induction (you may have already done this in the first step). This means it should look like:

$$\forall x, \ \big(x \text{ is a } \textsc{thing}\big) \implies P(x)$$

4) Write down a set of rules for generating all of the THINGs in your sentence. Each rule should involve inputs and outputs; make clear what these are.

5) For each rule A that you wrote, create a sentence that says

   > If $P(m)$ is true for all of the inputs m to Rule A then $P(n)$ is true for all of the outputs n of Rule A.

   (In other words, you should substitute the meaning of P and Rule A into the sentence above.) Diagram these sentences.

6) Prove the theorem by proving all of the sentences you just constructed (not to be done in class).

Theorem 4. *If n is a positive integer then*

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Theorem 5. *Suppose that numbers A_n are generated by the following rules:*

(i) $A_0 = 2$.

(ii) $A_1 = 5$.

(iii) If n is an integer and $n \geq 2$ then $A_n = 5A_{n-1} - 6A_{n-2}$.

Then $A_n = 2^n + 3^n$.

Theorem 6. *Every integer that is ≥ 18 can be generated by the following rules:*

(i) 18 *is an integer and* $18 \geq 18$.

(ii) 19 *is an integer and* $19 \geq 18$.

(iii) 20 *is an integer and* $20 \geq 18$.

(iv) 21 *is an integer and* $21 \geq 18$.

(v) If x is an integer and $x \geq 18$ then $x + 4$ is an integer and $x + 4 \geq 18$.

Theorem 7. *The integers that can be generated from* 4, 7, *and addition consist of* 4, 7, 8, 11, 12, 14, 15, 16 *and every integer* ≥ 18.

Theorem 8. *Suppose that n and d are integers, with $d \neq 0$. There are unique integers q and r with the following properties:*

(i) $n = qd + r$, *and*

(ii) $0 \leq r < |d|$.

Theorem 9. *Every positive integer can be generated by repeated application of the following rules:*

(i) 1 *is a positive integer.*

(ii) If x is a positive integer then $2x$ is a positive integer.

(iii) If x is a positive integer then $2x + 1$ is a positive integer.