1 **Definition 1.** An integer x is called *even* if there is an integer z such that $x = 2z$.

2 An integer x is called *odd* if there is an integer z such that $x = 2z + 1$.

3 Two integers are said to have the same *parity* if they are both even or are both odd.

4 **Theorem 2.** *The sum of two even integers is an even integer.*

Proof. This was proved on Exploration 4. Q.E.D.

5 **Theorem 3.** *The sum of an odd integer and an even integer is odd.*

6 *First proof of Theorem 3.* Let x be an odd integer and let y be an even integer.

7 By definition of oddness, there is an integer z such that $x = 2z + 1$.

8 By definition of evenness, there is an integer w such that $y = 2w$.

9 Therefore

$$x + y = 2z + 1 + 2w = 2(z + w) + 1.$$

10 Therefore $x + y$ is not a multiple of 2.

11 This means $x + y$ is not even so it must be odd. Q.E.D.

12 *Second proof of Theorem 3.* Let x be an odd integer and let y be an even integer.

13 By definition of oddness, there is an integer z such that $x = 2z + 1$.

14 By definition of evenness, there is an integer w such that $y = 2w$.

15 Therefore

$$x + y = 2z + 1 + 2w = 2(z + w) + 1.$$

16 Since z and w are integers, $z + w$ is an integer.

17 Therefore $x + y$ is odd, by the definition of oddness. Q.E.D.

Question 4. Do you like one proof more than the other?

Question 5. Are there any points in these proofs that could be made clearer?

18 **Theorem 6.** *The sum of two odd integers is even.*

Proof. This was proved on Exploration 4. Q.E.D.

19 *First proof of Theorem 6.* Let x and y be any two odd integers.

20 Since x and y are odd, the definition of oddness says that there are integers z and w such that $x = 2z + 1$ and $y = 2w + 1$.

21 Therefore

$$x + y = (2z + 1) + (2w + 1) = 2z + 2w + 2 = 2(z + w + 1).$$

22 The quantity $z + w + 1$ is a sum of integers, so it is an integer.

23 Therefore $x + y$ is 2 times an integer.

24 Thus $x + y$ is even, by the definition of evenness. Q.E.D.

25 *Second proof of Theorem 6.* Let x and y be any two odd integers.

26 Since x and y are odd, the definition of oddness says that there are integers z and w such that $x = 2z + 1$ and $y = 2w + 1$.

27 Therefore

$$x + y = (2z + 1) + (2w + 1) = 2z + 2w + 2.$$

28 Since z is an integer, $2z$ is even; since w is an integer, $2w$ is even; and since 1 is an integer, 2 is even (because $2 = 2 \times 1$).

29 Therefore $x + y$ is a sum of three even integers, so it is even. Q.E.D.

Question 7. Do you like one proof better than the other?

Question 8. Are there any points in these proofs that could be made clearer?

Question 9. Put the following sentences in forms that are suitable for direct proof.

(i) If x and y are real numbers such that $xy = 0$ then $x = 0$ or $y = 0$.

(ii) Every prime number other than 2 is odd.

(iii) There is no multiple of 6 that is not a multiple of 3.

(iv) There is no largest integer.

(v) No integer is both even and odd.

Question 10. What assumptions are made *implicitly* in the definitions, theorems, and proofs above?

Theorem 11. *The product of an even integer and any integer is an even integer.*

Theorem 12. *The product of two odd integers is an odd integer.*