Midterm Exam: Part 2

Math 2001–002, Fall 2016

October 25, 2016

Definition. The sequence $a_1, a_2, a_3, \ldots$ *converges* to b if the following statement is true:

for every real number $\epsilon > 0$, there is an integer $N > 0$ such that, for all $n > N$, we have $|a_n - b| < \epsilon$.

Instruction. The next two questions ask you to diagram mathematical sentences. In your diagrams you may use the symbols $\forall$, $\exists$, $\wedge$, $\vee$, $\Rightarrow$, $=$, $\neq$, $\leq$, $\geq$, $<$, $>$, variables, and the standard notation for arithmetic. *You may not use the word "not" or the negation symbol ($\neg$).*

Question 1. Diagram the sentence

The sequence $a_1, a_2, a_3, \ldots$ converges to b.

using the definition of convergence given above.

Question 2. Diagram the sentence

The sequence $a_1, a_2, a_3, \ldots$ does not converge to b.

using the definition of convergence given above. (Hint: You may want to use $x > y$ to mean the logical opposite of $x \leq y$.)

Question 3. The following definitions are used in this question:

Definition. An integer x is said to be a multiple of an integer y if there is an integer z such that $x = yz$.

Definition. A positive integer p is said to be *prime* if $p > 1$ and there are no integers d such that $1 < d < p$ and p is a multiple of d.

Put the sentences of the following proof in order by numbering them (1—15):

( ) Assume that n is an integer > 1 and that for every integer m such that $1 < m < n$ we know that m is divisible by a prime number. We wish to conclude from this that n is divisible by a prime number.

( ) By substitution, $n = pk\ell$.

( ) By the inductive hypothesis, a must be a multiple of a prime number p.

( ) **Theorem.** Every integer $n > 1$ is a multiple of a prime number.

( ) We will prove this by strong induction on n.

( ) If n is not prime then, as $n > 1$, by definition of primeness, there is an integer a such that $1 < a < n$ and n is a multiple of a.

( ) By definition, this means that n is a multiple of the prime number p.

( ) *Proof.*

( ) Therefore there is an integer ℓ such that $n = a\ell$.

( ) Therefore there is an integer c such that $n = cp$.

( ) Therefore there is an integer k such that $a = pk$.

( ) There are two possibilities: either n is prime or it is not prime.

( ) Since k and ℓ are integers, so is $k\ell$.

( ) If n is prime then it is divisible by a prime number because it is divisible by itself.

( ) Q.E.D.

Question 4. The Fibonacci sequence is defined by the following rules:

$$F_0 = 1$$
$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2} \qquad \text{for all integers } n \geq 2.$$

Warning! This is slightly different from the way the Fibonacci sequence was defined in the textbook.

Prove that the following formula is true for all integers $n \geq 0$:

$$F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1}.$$

Definition. An integer x is called...

...*threeven* if there is an integer y such that $x = 3y$,

...*throdd like* 1 if there is an integer y such that $x = 3y + 1$, and

...*throdd like* 2 if there is an integer y such that $x = 3y + 2$.

Question 5. Prove the following theorem:

Theorem. Prove that every integer is threeven or throdd like 1 or throdd like 2.

In your proof, make sure to reference the definition and use only basic facts about arithmetic.