Midterm Exam: Part 1

Math 2001–002, Fall 2016

October 25, 2016

Definition. An integer x is a *multiple* of an integer y if there is an integer z such that $x = yz$.

Instruction. The next two questions ask you to diagram mathematical sentences. In your diagrams you may use the symbols $\forall$, $\exists$, $\wedge$, $\Rightarrow$, $\vee$, $=$, $\neq$, variables, the standard notation for arithmetic, and phrases like "n is a ________". *You may not use the word "not" or the negation symbol ($\neg$).*

Question 1. Diagram the sentence "x is a multiple of y," interpreting *multiple* according to the definition above.

Question 2. Diagram the sentence "x is not a multiple of y" in a form suitable for *direct proof*, and interpreting *multiple* according to the definition above.

Question 3. The following definitions are used in this question:

> **Definition.** An integer x is said to be a multiple of an integer y if there is an integer z such that $x = yz$.
>
> **Definition.** A positive integer p is said to be *prime* if $p > 1$ and there are no integers d such that $1 < d < p$ and p is a multiple of d.

Put the sentences of the following proof in order by numbering them (1—15):

( ) Assume that n is an integer > 1 and that for every integer m such that $1 < m < n$ we know that m is divisible by a prime number. We wish to conclude from this that n is divisible by a prime number.

( ) By substitution, $n = pk\ell$.

( ) By the inductive hypothesis, a must be a multiple of a prime number p.

( ) **Theorem.** Every integer $n > 1$ is a multiple of a prime number.

( ) We will prove this by strong induction on n.

( ) If n is not prime then, as $n > 1$, by definition of primeness, there is an integer a such that $1 < a < n$ and n is a multiple of a.

( ) By definition, this means that n is a multiple of the prime number p.

( ) *Proof.*

( ) Therefore there is an integer ℓ such that $n = a\ell$.

( ) Therefore there is an integer c such that $n = cp$.

( ) Therefore there is an integer k such that $a = pk$.

( ) There are two possibilities: either n is prime or it is not prime.

( ) Since k and ℓ are integers, so is $k\ell$.

( ) If n is prime then it is divisible by a prime number because it is divisible by itself.

( ) Q.E.D.

Definition. An integer x is called...

...*threeven* if there is an integer y such that $x = 3y$,

...*throdd like* 1 if there is an integer y such that $x = 3y + 1$, and

...*throdd like* 2 if there is an integer y such that $x = 3y + 2$.

Question 4. Prove the following theorem:

Theorem. If a and b are any integers that are both throdd like 2 then ab is throdd like 1.

In your proof, make sure to reference the definition and use only basic facts about arithmetic.

Question 5. The Fibonacci sequence is defined by the following rules:

$$F_0 = 1$$
$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2} \qquad \text{for all integers } n \geq 2.$$

Warning! This is slightly different from the way the Fibonacci sequence was defined in the textbook.

Prove that the following formula is true for all integers $n \geq 0$:

$$F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1}.$$