CHAPTER

1 Fundamentals

The cornerstones of mathematics are definition, theorem, and proof. *Definitions* specify precisely the concepts in which we are interested, *theorems* assert exactly what is true about these concepts, and *proofs* irrefutably demonstrate the truth of these assertions.

Before we get started, though, we ask a question: Why?

1 Joy

Why?

Please also read the *To the Student* preface, where we briefly address the questions: What is mathematics, and what is discrete mathematics? We also give important advice on how to read a mathematics book.

Before we roll up our sleeves and get to work in earnest, I want to share with you a few thoughts on the question: Why study mathematics?

Mathematics is incredibly useful. Mathematics is central to every facet of modern technology: the discovery of new drugs, the scheduling of airlines, the reliability of communication, the encoding of music and movies on CDs and DVDs, the efficiency of automobile engines, and on and on. Its reach extends far beyond the technical sciences. Mathematics is also central to all the social sciences, from understanding the fluctuations of the economy to the modeling of social networks in schools or businesses. Every branch of the fine arts—including literature, music, sculpture, painting, and theater—has also benefited from (or been inspired by) mathematics

Because mathematics is both flexible (new mathematics is invented daily) and rigorous (we can incontrovertibly prove our assertions are correct), it is the finest analytic tool humans have developed.

The unparalleled success of mathematics as a tool for solving problems in science, engineering, society, and the arts is reason enough to actively engage this wonderful subject. We mathematicians are immensely proud of the accomplishments that are fueled by mathematical analysis. However, for many of us, this is not the primary motivation to study mathematics.

The Agony and the Ecstasy

Why do mathematicians devote their lives to the study of mathematics? For most of us, it is because of the joy we experience when doing mathematics.

Mathematics is difficult for everyone. No matter what level of accomplishment or skill in this subject you (or your instructor) have attained, there is always a harder, more frustrating problem waiting around the bend. Demoralizing? Hardly! The greater the challenge, the greater the sense of accomplishment we experience when the challenge has been met. The best part of mathematics is the joy we experience in practicing this art.

Most art forms can be enjoyed by spectators. I can delight in a concert performed by talented musicians, be awestruck by a beautiful painting, or be deeply moved by literature. Mathematics, however, releases its emotional surge only for those who actually do the work.

I want you to feel the joy, too. So at the end of this short section there is a single problem for you to tackle. In order for you to experience the joy, **do not under any circumstances let anyone help you solve this problem**. I hope that when you first look at the problem, you do not immediately see the solution but, rather, have to struggle with it for a while. Don't feel bad: I've shown this problem to extremely talented mathematicians who did not see the solution right away. Keep working and thinking—the solution will come. My hope is that when you solve this puzzle, it will bring a smile to your face. Here's the puzzle:

Conversely, if you have solved this problem, do not offer your assistance to others; you don't want to spoil their fun.

1 Exercise **1.1.** Simplify the following algebraic expression:

$$(x-a)(x-b)(x-c)\cdots(x-z).$$

2 Speaking (and Writing) of Mathematics

Precisely!

Whether or not we enjoy mathematics, we all can admire one of its unique features: there are definitive answers. Few other endeavors from economics to literary analysis to history to psychology can make this boast. Furthermore, in mathematics we can speak (and write) with extreme precision. While endless books, songs, and poems have been written about love, it's far easier to make precise statements (and verify their truth) about mathematics than human relations.

Precise language is vital to the study of mathematics. Unfortunately, students sometimes see mathematics as an endless series of numeric and algebraic calculations in which letters are only used to name variables; the closest one comes to using actual words is "sin" or "log."

In fact, to communicate mathematics clearly and precisely we need far more than numbers, variables, operations, and relation symbols; we need words composed into meaningful sentences that exactly convey the meaning we intend. Mathematical sentences often include technical notation, but the rules of grammar apply fully. Arguably, until one expresses ideas in a coherent sentence, those ideas are only half baked.

In addition, the mental effort to convert mathematical ideas into language is vital to learning those concepts. Take the time to express your ideas clearly both verbally and in writing. To learn mathematics requires you to engage all routes into your brain: your hands, eyes, mouth, and ears all need to get in on the act. Say the ideas out loud and write them down. You will learn to express yourself more clearly and you will learn the concepts better.

A Bit of Help

Be sure to check with your instructor concerning what types of collaboration are permitted on your assignments.

Writing is difficult. The best way to learn is to practice, especially with the help of a partner. Most people find it difficult to edit their own writing; our brains know what we want to say and trick us into believing that what we put onto paper is exactly what we intend. If you resort to saying "well, you know what I mean" then you need to try again.

In this brief section we provide a few pointers and some warnings about some common mistakes.

- *A language of our own*. Scattered in the margins of this book you will find *Mathspeak* notes that explain some of the idiosyncratic ways in which mathematicians use ordinary words. Common words (such as *function* or *prime*) are used differently in mathematics than in general use. The good news is that when we co-opt words into mathematical service, the meanings we give them are razor sharp (see the next section of this book for more about this).
- *Complete sentences*. This is the most basic rule of grammar and it applies to mathematics as much as to any discipline. Mathematical notation must be part of a sentence.

  Bad: $3x + 5$.

  This is not a sentence! What about $3x + 5$? What is the writer trying to say?

  Good: When we substitute $x = -5/3$ into $3x + 5$ the result is 0.

- *Mismatch of categories*. This is one of the most common mistakes people commit in mathematical writing and speaking. A line segment isn't a number, a function isn't an equation, a set isn't an operation, and so on. Consider this sentence:

  Air Force One is the president of the United States.

  This, of course, is nonsense. No amount of "well, you know what I mean" or "you get the general idea" can undo the error of writing that an airplane is a human being. Yet, this is exactly the sort of error novice mathematical writers (and speakers) make frequently.

  Thus, don't write "the function is equal to 3" when you mean "when the function is evaluated at $x = 5$ the result is 3". Note that we don't have to be verbose. Don't write "$f = 3$", but do write "$f(5) = 3$."

  Bad: If the legs of a right triangle T have lengths 5 and 12, then $T = 30$.

  Good: If the legs of a right triangle T have lengths 5 and 12, then the area of T is 30.
- *Avoid pronouns*. It's easy to write a sentence full of pronouns that you—the writer—understand but which is incomprehensible to anyone else.

  Bad: If we move everything over, then it simplifies and that's our answer.

  Give the things you're writing about names (such as single letters for numbers and line numbers for equations).

  Good: When we move all terms involving x to the left in equation (12), we find that those terms cancel and that enables us to determine the value of y.
- *Rewrite*. It's nearly impossible to write well on a first draft. What's more, few mathematics problems can be solved correctly straight away. Unfortunately, some students (not you, of course) start solving a problem, cross out errors, draw arrows to new parts of the solution, and then submit this awful mess as a finished product. Yuck! As with all other forms of writing, compose a first draft, edit, and then rewrite.
- *Learn LaTeX*. The editing and rewriting process is made much easier by word processors. Unfortunately, it's much more difficult to type mathematics than ordinary prose. Some what-you-see-is-what-you-get [WYSIWYG] word processing programs, such as Microsoft Word, include an equation editor that allows the typist to insert mathematical formulas into documents. Indeed, many scientists and engineers use Word to compose technical papers replete with intricate formulas.

  Nevertheless, the gold standard for mathematical typing is LaTeX. Learning to compose documents in LaTeX takes a significant initial investment of time, but no investment of cash as there many implementations of LaTeX that are free and run on most types of computers (Windows, Mac OS, linux). Documents produced in LaTeX are visually more appealing than the output of WYSIWIG systems and are easier to edit. In LaTeX one types special commands to produce mathematical notation. For example, to produce the quadratic formula

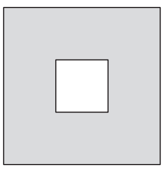
  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  one types: `x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}`

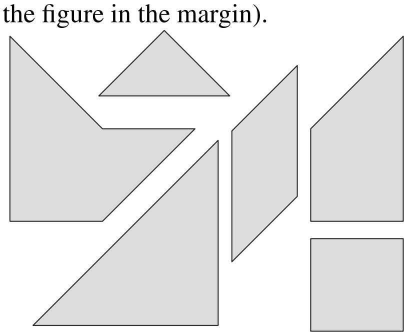
  There are many guides and books available for learning LaTeX including some that are available for free on the web.

The word LaTeX is written with letters of various sizes on different levels, in part to distinguish it from latex, a type of rubber. Incidentally, this book was composed using LaTeX.

2 Exercise



2.1. The six pieces below can be arranged to form a 3×3 square with the middle 1×1 square left empty (as in the figure in the margin).



Determine how to solve this puzzle and then write out clear instructions (without any diagrams!) so that another person can read your directions and properly fit the pieces together to arrive at the solution.

You may download a large, printable version of the puzzle pieces (so you can cut them out) from the author's website:

www.ams.jhu.edu/∼ers/mdi/puzzle.pdf

3 Definition

Mathematics exists only in people's minds. There is no such "thing" as the number 6. You can draw the symbol for the number 6 on a piece of paper, but you can't physically hold a 6 in your hands. Numbers, like all other mathematical objects, are purely conceptual.

Mathematical objects come into existence by definitions. For example, a number is called *prime* or *even* provided it satisfies precise, unambiguous conditions. These highly specific conditions are the definition for the concept. In this way, we are acting like legislators, laying down specific criteria such as eligibility for a government program. The difference is that laws may allow for some ambiguity, whereas a mathematical definition must be absolutely clear.

Let's take a look at an example.

Definition 3.1 **(Even)** An integer is called *even* provided it is divisible by two.

In a definition, the word(s) being defined are typically set in *italic* type.

Clear? Not entirely. The problem is that this definition contains terms that we have not yet defined, in particular *integer* and *divisible*. If we wish to be terribly fussy, we can complain that we haven't defined the term *two*. Each of these terms—*integer*, *divisible*, and *two*—can be defined in terms of simpler concepts, but this is a game we cannot entirely win. If every term is defined in terms of simpler terms, we will be chasing definitions forever. Eventually we must come to a point where we say, "This term is undefined, but we think we understand what it means."

The situation is like building a house. Each part of the house is built up from previous parts. Before roofing and siding, we must build the frame. Before the frame goes up, there must be a foundation. As house builders, we think of pouring the foundation as the first step, but this is not really the first step. We also have to own the land and run electricity and water to the property. For there to be water, there must be wells and pipes laid in the ground. STOP! We have descended to a level in the process that really has little to do with building a house. Yes, utilities are vital to home construction, but it is not our job, as home builders, to worry about what sorts of transformers are used at the electric substation!

Let us return to mathematics and Definition 3.1. It is possible for us to define the terms *integer*, *two*, and *divisible* in terms of more basic concepts. It takes a great deal of work to define integers, multiplication, and so forth in terms of simpler concepts. What are we to do? Ideally, we should begin from the most basic mathematical object of all—the *set*—and work our way up to the integers. Although this is a worthwhile activity, in this book we build our mathematical house assuming the foundation has already been laid.

The symbol $\mathbb{Z}$ stands for the integers. This symbol is easy to draw, but often people do a poor job. Why? They fall into the following trap: They first draw a Z and then try to add an extra slash. That doesn't work! Instead, make a 7 and then an interlocking, upside-down 7 to draw $\mathbb{Z}$.

Where shall we begin? What may we assume? In this book, we take the integers as our starting point. The *integers* are the positive whole numbers, the negative whole numbers, and zero. That is, the set of integers, denoted by the letter $\mathbb{Z}$, is

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

We also assume that we know how to add, subtract, and multiply, and we need not prove basic number facts such as $3 \times 2 = 6$. We assume the basic algebraic properties of addition, subtraction, and multiplication and basic facts about order relations ($<$, $\leq$, $>$, and $\geq$). See Appendix D for more details on what you may assume.

Thus, in Definition 3.1, we need not define *integer* or *two*. However, we still need to define what we mean by *divisible*. To underscore the fact that we have not made this clear yet, consider the question: Is 3 divisible by 2? We want to say that the answer to this question is no, but perhaps the answer is yes since $3 \div 2$ is $1\frac{1}{2}$. So it is possible to divide 3 by 2 if we allow fractions. Note further that in the previous paragraph we were granted basic properties of addition, subtraction, and multiplication, but not—and conspicuous by its absence—division. Thus we need a careful definition of *divisible*.

Definition 3.2 **(Divisible)** Let a and b be integers. We say that a is *divisible* by b provided there is an integer c such that $bc = a$. We also say b *divides* a, or b is a *factor* of a, or b is a *divisor* of a. The notation for this is $b|a$.

This definition introduces various terms (*divisible*, *factor*, *divisor*, and *divides*) as well as the notation $b|a$. Let's look at an example.

Example 3.3 Is 12 divisible by 4? To answer this question, we examine the definition. It says that $a = 12$ is divisible by $b = 4$ if we can find an integer c so that $4c = 12$. Of course, there is such an integer, namely, $c = 3$.

In this situation, we also say that 4 divides 12 or, equivalently, that 4 is a factor of 12. We also say 4 is a divisor of 12.

The notation to express this fact is $4|12$.

On the other hand, 12 is not divisible by 5 because there is no integer x for which $5x = 12$; thus $5|12$ is false.

Now Definition 3.1 is ready to use. The number 12 is even because $2|12$, and we know $2|12$ because $2 \times 6 = 12$. On the other hand, 13 is not even, because 13 is not divisible by 2; there is no integer x for which $2x = 13$. Note that we did not say that 13 is odd because we have yet to define the term *odd*. Of course, we know that 13 is an odd number, but we simply have not "created" odd numbers yet by specifying a definition for them. All we can say at this point is that 13 is not even. That being the case, let us define the term *odd*.

Definition 3.4 **(Odd)** An integer a is called *odd* provided there is an integer x such that $a = 2x + 1$.

Thus 13 is odd because we can choose $x = 6$ in the definition to give $13 = 2 \times 6 + 1$. Note that the definition gives a clear, unambiguous criterion for whether or not an integer is odd.

Please note carefully what the definition of *odd* does not say: It does not say that an integer is odd provided it is not even. This, of course, is true, and we prove it in a subsequent chapter. "Every integer is odd or even but not both" is a fact that we *prove*.

Here is a definition for another familiar concept.

Definition 3.5 **(Prime)** An integer p is called *prime* provided that $p > 1$ and the only positive divisors of p are 1 and p.

For example, 11 is prime because it satisfies both conditions in the definition: First, 11 is greater than 1, and second, the only positive divisors of 11 are 1 and 11.

However, 12 is not prime because it has a positive divisor other than 1 and itself; for example, $3|12$, $3 \neq 1$, and $3 \neq 12$.

Is 1 a prime? No. To see why, take $p = 1$ and see if p satisfies the definition of primality. There are two conditions: First we must have $p > 1$, and second, the only positive divisors of p are 1 and p. The second condition is satisfied: the only divisors of 1 are 1 and itself. However, $p = 1$ does not satisfy the first condition because $1 > 1$ is false. Therefore, 1 is not a prime.

We have answered the question: Is 1 a prime? The reason why 1 isn't prime is that the definition was specifically designed to make 1 nonprime! However, the real "why question" we would like to answer is: Why did we write Definition 3.5 to exclude 1?

I will attempt to answer this question in a moment, but there is an important philosophical point that needs to be underscored. The decision to exclude the number 1 in the definition was deliberate and conscious. In effect, the reason 1 is not prime is "because I said so!" In principle, you could define the word *prime* differently and allow the number 1 to be prime. The main problem with your using a different definition for prime is that the concept of a *prime number* is well established in the mathematical community. If it were useful to you to allow 1 as a prime in your work, you ought to choose a different term for your concept, such as *relaxed prime* or *alternative prime*.

Now, let us address the question: Why did we write Definition 3.5 to exclude 1? The idea is that the prime numbers should form the "building blocks" of multiplication. Later, we prove the fact that every positive integer can be factored in a unique fashion into prime numbers. For example, 12 can be factored as $12 = 2 \times 2 \times 3$. There is no other way to factor 12 down to primes (other than rearranging the order of the factors). The prime factors of 12 are precisely 2, 2, and 3. Were we to allow 1 as a prime number, then we could also factor 12 down to "primes" as $12 = 1 \times 2 \times 2 \times 3$, a different factorization.

Since we have defined prime numbers, it is appropriate to define composite numbers.

Definition 3.6 **(Composite)** A positive integer a is called *composite* provided there is an integer b such that $1 < b < a$ and $b|a$.

For example, the number 25 is composite because it satisfies the condition of the definition: There is a number b with $1 < b < 25$ and $b|25$; indeed, $b = 5$ is the only such number.

Similarly, the number 360 is composite. In this case, there are several numbers b that satisfy $1 < b < 360$ and $b|360$.

Prime numbers are not composite. If p is prime, then, by definition, there can be no divisor of p between 1 and p (read Definition 3.5 carefully).

Furthermore, the number 1 is not composite. (Clearly, there is no number b with $1 < b < 1$.) Poor number 1! It is neither prime nor composite! (There is, however, a special term that is applied to the number 1—the number 1 is called a *unit*.)

Recap

In this section, we introduced the concept of a mathematical definition. Definitions typically have the form "An object X is called *the term being defined* provided it satisfies *specific conditions*." We presented the integers $\mathbb{Z}$ and defined the terms *divisible*, *odd*, *even*, *prime*, and *composite*.

3 Exercises

3.1. Please determine which of the following are true and which are false; use Definition 3.2 to explain your answers.
- **a.** $3|100$.
- **b.** $3|99$.
- **c.** $-3|3$.
- **d.** $-5|-5$.
- **e.** $-2|-7$.
- **f.** $0|4$.
- **g.** $4|0$.
- **h.** $0|0$.

3.2. Here is a possible alternative to Definition 3.2: We say that a is *divisible* by b provided $\frac{a}{b}$ is an integer. Explain why this alternative definition is different from Definition 3.2.

Here, *different* means that Definition 3.2 and the alternative definition specify *different concepts*. So, to answer this question, you should find integers a and b such that a is divisible by b according to one definition, but a is not divisible by b according to the other definition.

3.3. None of the following numbers is prime. Explain why they fail to satisfy Definition 3.5. Which of these numbers is composite?
- **a.** 21.
- **b.** 0.
- **c.** π.
- **d.** $\frac{1}{2}$.
- **e.** -2.
- **f.** -1.

The symbol $\mathbb{N}$ stands for the natural numbers.

3.4. The *natural numbers* are the nonnegative integers; that is,

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

Use the concept of natural numbers to create definitions for the following relations about integers: *less than* ($<$), *less than or equal to* ($\leq$), *greater than* ($>$), and *greater than or equal to* ($\geq$).

Note: Many authors define the natural numbers to be just the positive integers; for them, zero is not a natural number. To me, this seems unnatural ☺. The concepts *positive integers* and *nonnegative integers* are unambiguous and universally recognized among mathematicians. The term *natural number*, however, is not 100% standardized.

The symbol $\mathbb{Q}$ stands for the rational numbers.

3.5. A *rational number* is a number formed by dividing two integers a/b where $b \neq 0$. The set of all rational numbers is denoted $\mathbb{Q}$.

Explain why every integer is a rational number, but not all rational numbers are integers.

3.6. Define what it means for an integer to be a *perfect square*. For example, the integers 0, 1, 4, 9, and 16 are perfect squares. Your definition should begin

An integer x is called a *perfect square* provided. . . .

3.7. Define what it means for one number to be the *square root* of another number.

3.8. Define the *perimeter* of a polygon.

3.9. Suppose the concept of distance between points in the plane is already defined. Write a careful definition for one point to be *between* two other points. Your definition should begin

Suppose A, B, C are points in the plane. We say that C is *between* A and B provided. . . .

Note: Since you are crafting this definition, you have a bit of flexibility. Consider the possibility that the point C might be the same as the point A or B, or even that A and B might be the same point. Personally, if A and C were the same point, I would say that C is between A and B (regardless of where B may lie), but you may choose to design your definition to exclude this possibility. Whichever way you decide is fine, but be sure your definition does what you intend.

Note further: You do not need the concept of collinearity to define *between*. Once you have defined *between*, please use the notion of between to define what it means for three points to be collinear. Your definition should begin

Suppose A, B, C are points in the plane. We say that they are collinear provided. . . .

Note even further: Now if, say, A and B are the same point, you certainly want your definition to imply that A, B, and C are collinear.

3.10. Define the *midpoint* of a line segment.

3.11. Some English words are difficult to define with mathematical precision (for example, *love*), but some can be tightly defined. Try writing definitions for these:

a. teenager.
b. grandmother.
c. leap year.
d. dime.
e. palindrome.
f. homophone.

You may assume more basic concepts (such as *coin* or *pronunciation*) are already defined.

3.12. Discrete mathematicians especially enjoy *counting problems*: problems that ask *how many*. Here we consider the question: How many positive divisors does a number have? For example, 6 has four positive divisors: 1, 2, 3, and 6.

How many positive divisors does each of the following have?

a. 8.
b. 32.
c. 2^n where n is a positive integer.
d. 10.
e. 100.

f. 1,000,000.
g. 10^n where n is a positive integer.
h. $30 = 2 \times 3 \times 5$.
i. $42 = 2 \times 3 \times 7$. (Why do 30 and 42 have the same number of positive divisors?)
j. $2310 = 2 \times 3 \times 5 \times 7 \times 11$.
k. $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8$.
l. 0.

3.13. An integer n is called *perfect* provided it equals the sum of all its divisors that are both positive and less than n. For example, 28 is perfect because the positive divisors of 28 are 1, 2, 4, 7, 14, and 28. Note that $1 + 2 + 4 + 7 + 14 = 28$.
a. There is a perfect number smaller than 28. Find it.
b. Write a computer program to find the next perfect number after 28.

3.14. *At a Little League game there are three umpires. One is an engineer, one is a physicist, and one is a mathematician. There is a close play at home plate, but all three umpires agree the runner is out.*

Furious, the father of the runner screams at the umpires, "Why did you call her out?!"

The engineer replies, "She's out because I call them as they are."
The physicist replies, "She's out because I call them as I see them."
The mathematician replies, "She's out because I called her out."
Explain the mathematician's point of view.

4 Theorem

A *theorem* is a declarative statement about mathematics for which there is a proof.

The notion of proof is the subject of the next section—indeed, it is a central theme of this book. Suffice it to say for now that a *proof* is an essay that incontrovertibly shows that a statement is true.

In this section we focus on the notion of a theorem. Reiterating, a *theorem* is a declarative statement about mathematics for which there is a proof.

What is a declarative statement? In everyday English we utter many types of sentences. Some sentences are questions: Where is the newspaper? Other sentences are commands: Come to a complete stop. And perhaps the most common sort of sentence is a *declarative statement*—a sentence that expresses an idea about how something is, such as: It's going to rain tomorrow or The Yankees won last night.

Practitioners of every discipline make declarative statements about their subject matter. The economist says, "If the supply of a commodity decreases, then its price will increase." The physicist asserts, "When an object is dropped near the surface of the earth, it accelerates at a rate of 9.8 meter/sec^2."

Mathematicians also make statements that we believe are true about mathematics. Such statements fall into three categories:

- Statements we know to be true because we can prove them—we call these *theorems*.
- Statements whose truth we cannot ascertain—we call these *conjectures*.
- Statements that are false—we call these *mistakes*!

Please be sure to check your own work for nonsensical sentences. This type of mistake is all too common. Think about every word and symbol you write. Ask yourself, what does this term mean? Do the expressions on the left and right sides of your equations represent objects of the same type?

There is one more category of mathematical statements. Consider the sentence "The square root of a triangle is a circle." Since the operation of extracting a square root applies to numbers, not to geometric figures, the sentence doesn't make sense. We therefore call such statements *nonsense*!

The Nature of Truth

To say that a statement is *true* asserts that the statement is correct and can be trusted. However, the nature of truth is much stricter in mathematics than in any other discipline. For example, consider the following well-known meteorological fact: "In July, the weather in Baltimore is