Problem 1. Prove that for every finite n with $n \geq 2$,

$$\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}.$$

Solution. Let S be a set with $n \geq 2$ elements. Choose two elements, x and y. Consider the function

$$f : \binom{S}{k} \to 2^{\{x,y\}}$$

defined by the formula $f(T) = T \cap \{x, y\}$. We have:

$$f^{-1}\{\varnothing\} = \binom{S - \{x,y\}}{k}$$

We have a bijection

$$u : f^{-1}\{\{x\}\} = \{T \in \binom{S}{k} \,:\, x \in T\} \to \binom{S - \{x,y\}}{k-1}$$
$$u(T) = T \smallsetminus \{x\}.$$

We have a bijection

$$v : f^{-1}\{\{y\}\} = \{T \in \binom{S}{k} \,:\, y \in T\} \to \binom{S - \{x,y\}}{k-1}$$
$$v(T) = T \smallsetminus \{y\}.$$

Finally, we have a bijection

$$w : f^{-1}\{\{x,y\}\} = \{T \in \binom{S}{k} \,:\, x \in T \wedge y \in T\} \to \binom{S - \{x,y\}}{k-2}$$
$$w(T) = T \smallsetminus \{x, y\}.$$

Thus

$$\left|\binom{S}{k}\right| = \left|\binom{S \smallsetminus \{x,y\}}{k}\right| + \left|\binom{S \smallsetminus \{x\}}{k-1}\right| + \left|\binom{S \smallsetminus \{y\}}{k-1}\right| + \left|\binom{S \smallsetminus \{x,y\}}{k-2}\right|$$

which is the desired formula. □

Problem 2. Let S be set of size n. Denote by $\binom{S}{a\ b}$ the set of all lists (A, B) where $\{A, B\}$ is a partition of S such that $|A| = a$ and $|B| = b$. Which of the following are equal to $|\binom{S}{a\ b}|$?

A) $\binom{n}{a}$
B) $\binom{n}{b}$
C) $\frac{1}{2}\binom{n}{a}$
D) More than one of the above
E) None of the above

Solution. D) □

Problem 3. Let $f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k$. Which of the following agrees with $f(1)$ for all natural numbers n?

A) 1 B) $\binom{n+1}{k+1}$ C) $\binom{n}{n/2}$ D) 2^n E) 3^n

Solution. D) □

Problem 4. Let $f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k$. Which of the following agrees with $f(2)$ for all natural numbers n?

A) 1 B) $\binom{n+1}{k+1}$ C) $\binom{n}{n/2}$ D) 2^n E) 3^n

Solution. E) □

Problem 5. Prove that $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ for all natural numbers n.

Solution. By induction on n. If $n = 0$ then $(x+y)^n = 1$ and

$$\sum_{k=0}^{0} \binom{0}{k} x^k = \binom{0}{0} x^0 = 1$$

so the formula works when $n = 0$. Supposing the formula holds for n we prove it also holds for $n+1$. We have

$$\begin{aligned}
(1+x)^{n+1} &= (1+x)(1+x)^n = (x+y) \sum_{k=0}^{n} \binom{n}{k} x^k \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k + \sum_{k=0}^{n} \binom{n}{k} x^{k+1} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k + \sum_{k=1}^{n+1} \binom{n}{k-1} x^k \\
&= \sum_{k=0}^{n+1} \binom{n}{k} x^k + \sum_{k=0}^{n+1} \binom{n}{k-1} x^k \\
&= \sum_{k=0}^{n+1} \left(\binom{n}{k} x^k + \binom{n}{k-1} x^k \right) \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k.
\end{aligned}$$

□