Problem 1. Suppose that S is a set of size n and $0 \leq k \leq n$. How many subsets of size k does S have?

Solution. This time we will develop an inductive approach to calculation. If $|S| = 0$ then $S = \varnothing$ and S has exactly one subset of size 0 and no subsets of any other sizes. If $|S| > 0$ then we can pick an element $x \in S$. Notice that $\binom{S}{k}$ is the disjoint union of two subsets: Let $A \subset \binom{S}{k}$ be the set of $U \in \binom{S}{k}$ such that $x \in U$ and let $B \subset \binom{S}{k}$ be the set of $U \in \binom{S}{k}$ such that $x \notin U$. Then $\{A, B\}$ is a partition of $\binom{S}{k}$ so

$$\left|\binom{S}{k}\right| = |A| + |B|.$$

Of course, $B = \binom{S-\{x\}}{k}$. We can also construct a bijection between A and $\binom{S-\{x\}}{k-1}$. Define $f : \binom{S-\{x\}}{k-1} \to A$ and $g : A \to \binom{S-\{x\}}{k-1}$ by the formulas below:

$$f(U) = U \cup \{x\} g(V) = V \smallsetminus \{x\}.$$

Thus

$$\left|\binom{S}{k}\right| = \left|\binom{S-\{x\}}{k-1}\right| + \left|\binom{S-\{x\}}{k}\right|.$$

□

Solution. Here is an approach using a function. If $|S| = 0$, the number is 1. If $|S| > 0$, pick $x \in S$. Then we have a function

$$f : \binom{S}{k} \to 2^{\{x\}}$$

defined by $f(U) = U \cap \{x\}$. Then $f^{-1}\{\varnothing\} = \binom{S-\{x\}}{k}$ and $f^{-1}\{\{x\}\}$ is in bijection with $\binom{S-\{x\}}{k-1}$. Therefore

$$\left|\binom{S}{k}\right| = \left|\binom{S-\{x\}}{k-1}\right| + \left|\binom{S-\{x\}}{k}\right|.$$

□