Definition 1. If S is a set, we write 2^S for the set of subsets of S and we write $\binom{S}{k}$ for the set of subsets of S of size k.

Problem 2. Compute $\left|\binom{S}{2}\right|$ when $S = \{1, 2, 3, 4\}$.
A) 0 B) 2 C) 3 D) 4 E) 6

Solution. E) □

Problem 3. Compute $\left|\binom{S}{5}\right|$ when $S = \{1, 2, 3, 4\}$.
A) 0 B) 2 C) 3 D) 4 E) 6

Solution. A) □

Problem 4. Suppose S is a set with n elements and k is an integer with $0 \leq k \leq n$. How many subsets of size k does S have?

(i) Choose a set S with 4 elements. Write down the set of all subsets of S of size 2. How many elements does it have? Write down the set of all *lists* of length 2 drawn from S *without repetition*. How many elements does it have?

(ii) Repeat the last part with subsets of size 3 and lists of length 3 without repetition.

(iii) Find and prove a general formula.

Solution. Let U be the set of all subsets of S of size k. We want to count the number of elements of U.

Let T be the set of all lists of length k drawn from S without repetition. We know that T has $(n)_k = \frac{n!}{(n-k)!}$ elements. We have a function:

$$p : T \to U$$
$$p((x_1, x_2, \ldots, x_k)) = \{x_1, x_2, \ldots, x_k\}$$

Since we know the size of T, we should be able to figure out the size of U if we understand how many elements of T go to each element of U. To figure this out, notice that for any $u \in U$, the set

$$\{t \in T \ : \ p(t) = u\}$$

is the set of all lists of length k drawn from u! There are exactly $(k)_k = k!$ of these. Therefore, for each element $u \in U$ there are $k!$ elements of T. We can write this in the formula

$$|T| = k! \times |U|.$$

Since we know that $|T| = \frac{n!}{(n-k)!}$, this gives us a formula for the size of U:

$$|U| = \frac{n!}{(n-k)!} \times \frac{1}{k!} = \frac{n!}{k!(n-k)!}.$$

□

Definition 5. If $f : T \to U$ is any function and $V \subset U$ is a subset then the *pre-image* of V in T is the set

$$f^{-1}V = \{t \in T \ : \ f(t) \in V\}.$$

Problem 6. Let $f : \mathbf{R} \to \mathbf{R}$ be the function $f(x) = x^2$. Compute $f^{-1}\{4\}$.
A) $\varnothing$ B) 2 C) $\{2\}$ D) $\{-2, 2\}$ E) $\mathbf{R}$

Solution. D) □

Problem 7. Let $f : \mathbf{R} \to \mathbf{R}$ be the function $f(x) = x^2$. Compute $f^{-1}\{-4\}$.
A) $\varnothing$ B) 2 C) $\{2\}$ D) $\{-2, 2\}$ E) $\mathbf{R}$

Solution. A) $\square$

Theorem 8. *If $f : T \to U$ is any function then*

$$|T| = \sum_{u \in U} |f^{-1}\{u\}|.$$

Proof. We show that the sets $f^{-1}\{u\}$ are pairwise disjoint with union T. First we show that the union is T. Suppose that $t \in T$. Let $u = f(t)$. Then $t \in f^{-1}\{u\}$, by definition. Therefore $t \in \bigcup_{u \in U} f^{-1}\{u\}$. This applies to all $t \in T$, so $T \subset \bigcup_{u \in U} f^{-1}\{u\}$. On the other hand, by definition $\bigcup_{u \in U} f^{-1}\{u\}$ is a union of subsets of T, hence is a subset of T, to $\bigcup_{u \in U} f^{-1}\{u\} \subset T$.

Now we show that the sets $f^{-1}\{u\}$ are pairwise disjoint. Suppose that $x \in f^{-1}\{u\} \cap f^{-1}\{v\}$. Then $f(x) = u$ and $f(x) = v$. But f is a function, so each input has a unique output. In particular, $u = v$ and therefore $f^{-1}\{u\} = f^{-1}\{v\}$.

Now,

$$|T| = \sum_{t \in T} 1 = \sum_{u \in U} \sum_{t \in f^{-1}\{u\}} 1 = \sum_{u \in U} |f^{-1}\{u\}|.$$

The reason the second equality is true is because each $t \in T$ is in $f^{-1}\{u\}$ for exactly one $u \in U$. $\square$