Definition 1. Let A and B be sets. A relation f from A to B...

is called...	if...
a **function**	$\forall\, a \in A,\ \exists!\, b \in B,\ (a,b) \in f$
injective or **one-to-one**	$\forall\, a, a' \in A, \forall\, b \in B,\ \Big((a,b) \in f \wedge (a',b) \in f \implies a = a'\Big)$
surjective or **onto**	$\forall\, b \in B,\ \exists\, a \in A,\ (a,b) \in f$
bijective	f is injective and surjective
an **injection**	f is an injective function
a **surjection**	f is a surjective function
a **bijection**	f and f^{-1} are both functions

The *image* of f is

$$\operatorname{im} f = \{b \in B\ :\ \exists\, a \in A,\ (a,b) \in f\}.$$

If f is a function from A to B then and $a \in A$ then we write $f(a)$ for the unique $b \in B$ such that $(a,b) \in f$. We write $f : A \to B$ to mean "f is a function from A to B." We call A the *domain* of f and B the *codomain* of f.

Problem 2. Let f be the function given by $f(x) = x^2$. Which of the following are true?

A) f is injective.

B) f is surjective.

C) f is both injective and surjective.

D) f is neither injective nor surjective.

E) What are the domain and codomain?

Solution. E) □

Problem 3. Let $f : \mathbf{R} \to \mathbf{R}$ be the function given by $f(x) = x^2$. Which of the following are true?

A) f is injective.

B) f is surjective.

C) f is both injective and surjective.

D) f is neither injective nor surjective.

E) What are the domain and codomain?

Solution. D) □

Problem 4. Let $f : \mathbf{R} \to \mathbf{R}_{\geq 0}$ be the function given by $f(x) = x^2$. Which of the following are true?

A) f is injective.

B) f is surjective.

C) f is both injective and surjective.

D) f is neither injective nor surjective.

E) What are the domain and codomain?

Solution. B) □

Problem 5. Let $f : \mathbf{R}_{\geq 0} \to \mathbf{R}_{\geq 0}$ be the function given by $f(x) = x^2$. Which of the following are true?

A) f is injective.

B) f is surjective.

C) f is both injective and surjective.

D) f is neither injective nor surjective.

E) What are the domain and codomain?

Solution. C) ☐

Problem 6. Let S be a set and let $f : 2^S \to 2^S$ be the function

$$f(T) = S - T.$$

Which of the following are true?

A) f is injective.

B) f is surjective.

C) f is both injective and surjective.

D) f is neither injective nor surjective.

Solution. C) ☐

Theorem 7. *Suppose f is a function from A to B. The following properties are equivalent:*

(i) f is both injective and surjective from A to B;

(ii) f^{-1} is a function from B to A.

Proof. Suppose that f is both injective and surjective from A to B. We prove that f^{-1} is a function from B to A. To show this, we need to show that, for every $b \in B$ there is a unique $a \in A$ such that $(b, a) \in f^{-1}$. Suppose that $b \in B$. By the surjectivity of f, there is at least one $a \in A$ such that $(a, b) \in f$. By definition of f^{-1}, this means $(b, a) \in f^{-1}$. To see that a is unique, suppose that $(b, a') \in f^{-1}$. Then $(a', b) \in f$. But f is injective, and both (a, b) and $(a', b) \in f$. Therefore $a = a'$. This shows that a is unique.

Let us now suppose that f^{-1} is a function from B to A. To see that f is surjective, we need to show that, for every $b \in B$, there is at least one $a \in A$ such that $(a, b) \in f$. Let $a = f^{-1}(b)$. Then by definition, we have $(b, a) \in f^{-1}$. By definition of f^{-1}, this means that $(a, b) \in f$. Thus $f(a) = b$, by definition. This proves f is surjective.

To show that f is injective, we must show that if $f(a) = f(a')$ then $a = a'$. Suppose that $a \in A$ and that $f(a) = f(a')$. Let's write $b = f(a) = f(a')$ for brevity. Then $(a, b) \in f$ and $(a', b) \in f$. By definition of f^{-1}, this means that $(b, a) \in f^{-1}$ and $(b, a') \in f^{-1}$. That is, $a = f^{-1}(b)$ and $a' = f^{-1}(b)$. Therefore $a = a'$. This proves f is injective. ☐

Definition 8. Suppose A and B are sets. We write $|A| \leq |B|$ and say that the *cardinality* of A is less than or equal to the cardinality of B if there is an injection from A to B. We write that $|A| = |B|$ and say that A and B have the same cardinality of there is a bijection from A to B.