1 Proof techniques

Proof Technique 1 (Contraposition)**.** To prove a statement of the form

$$P \implies Q$$

instead prove the statement

$$(\neg Q) \implies (\neg P).$$

Problem 1.1. Prove that, for any real numbers x and y, if $xy = 0$ then $x = 0$ or $y = 0$.

Proof Technique 2 (Induction)**.** To prove a statement of the form

$$\forall\, n \in \mathbb{N},\ P(n)$$

prove the following statements:

(i) $P(0)$

(ii) $\forall\, n \in \mathbb{N},\ (P(n) \implies P(n+1))$

Problem 2.1. Prove that $2^n < 3^n$ for every integer $n > 0$.

Problem 2.2. Suppose that p and q are positive real numbers such that $p < q$. Prove that $p^n < q^n$ for all natural numbers $n > 1$. You may use the following facts about the $<$ relation: If $x < y$ and $0 < a < b$ then $ax < by$.

Proof Technique 3 (Proof by cases)**.** To prove a statement

$$(P \vee Q) \implies R$$

prove the two statements

$$P \implies R$$
$$Q \implies R.$$

Problem 3.1. Prove that for any natural number n either n or $n + 1$ is divisible by 2.

Proof Technique 4 (Strong induction)**.** To prove a statement of the form

$$\forall\, n \in \mathbb{N},\ P(n)$$

prove the sentence

$$\forall\, n \in \mathbb{N},\ \Big((\forall m \in \mathbb{N},\ (m < n \implies P(m))) \implies P(n) \Big).$$

Here is a slightly less precise but more readable version of this sentence:

$$\forall\, n,\ \Big((\forall\, m < n,\ P(m)) \implies P(n) \Big)$$

Problem 4.1. Prove that, for every integer n, either there is an integer m such that $nm = 1$ or there is a prime number p such that $p|n$.

Problem 4.2. Let $a_0, a_1, a_2, \ldots$ be the sequence with the following properties:

(i) $a_0 = a_1 = 0$

(ii) $a_2 = 1$

(iii) $a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3}$

Prove that $a_n = 2^n - n - 1$ for all natural numbers n.

Solution. We have $2^0 - 0 - 1 = 0$ and $2^1 - 1 - 1 = 0$ and $2^2 - 2 - 1 = 1$. This gives a base case.

Now assume for the sake of strong induction that $n \geq 3$ and that $a_m = 2^m - m - 1$ for $m < n$. We want to prove that $a_n - 2^n - n - 1 = 0$. We have

$$\begin{aligned}
a_n &= 4a_{n-1} - 5a_{n-2} + 2a_{n-3} \\
&= 4 \times (2^{n-1} - n) - 5 \times (2^{n-2} - n + 1) + 2 \times (2^{n-3} - n + 2) \\
&= 2 \times 2^n - 2^n - 2^{n-2} + 2^{n-2} - 4n + 5n - 2n - 5 + 4 \\
&= 2^n - n - 1
\end{aligned}$$

exactly as desired. This proves the induction step and completes the proof. □

Problem 4.3. Prove that, for any natural number n, one of the numbers n, $n+1$, or $n+2$ is divisible by 3.

Solution. The proof is by induction on n. Let $P(n)$ be the sentence "Either n, $n+1$, or $n+2$ is divisible by 3." The statement is certainly true for $n = 0$, since 0 is divisible by 3, so we have the base case for induction.

Assume that n is an integer such that $P(n)$ is true. We prove that $P(n+1)$ is true. We know from $P(n)$ that either n, $n+1$, or $n+2$ is divisible by 3. If $3|n+1$ or $3|n+2$ then $P(n+1)$ is obviously true (after all $P(n+1)$ is the sentence $3|n+1 \vee 3|n+2 \vee 3|n+3$). The only remaining possibility is that $3|n$. In that case $n = 3k$ for some integer k (by the definition of divisibility), so $n+3 = 3k+3 = 3(k+1)$. As $k+1$ is an integer, this tells us that $n+3$ is divisible by 3, again by the definition of divisibility. Thus

$$3|n \vee 3|n+1 \vee 3|n+2 \implies 3|n+1 \vee 3|n+2 \vee 3|n+3,$$

which proves the induction step. □

Problem 4.4. Prove by induction that every non-empty subset of $\mathbb{N}$ has a least element.

Solution. Suppose that $S \subset \mathbb{N}$ is a subset without a least element. We prove by strong induction that $S = \varnothing$. Let $P(n)$ be the sentence $n \notin S$. We prove $P(n)$ for all natural numbers n by strong induction on n.

Let n be a natural number and assume for the sake of strong induction that, for all $m < n$, the sentence $P(m)$ is true. That is, for all $m < n$, we know that $m \notin S$. Therefore, if n were in S then n would be the least element of S. As S has no least element, we conclude that $m \notin S$. Therefore $P(n)$ is also true. This proves the induction step, so by strong induction we conclude that $S = \varnothing$. □

Proof Technique 5 (Bidirectional induction)**.** To prove a statement of the form

$$\forall\, n \in \mathbb{Z},\ P(n)$$

instead prove the two sentences

(i) $\exists\, n \in \mathbb{Z},\ P(n)$

(ii) $\forall\, n \in \mathbb{Z}, (P(n) \iff P(n+1))$

2 Other problems

Problem 2.1. Suppose that $A = \{a, b, c\}$ and R is a total order on R such that

$$\begin{gathered}
(a, b) \in R \\
(b, c) \in R \\
(c, a) \in R.
\end{gathered}$$

What is the cardinality of A? Justify your answer.

Problem 2.2. Prove that $1 + 2 + 4 + 8 + \cdots + 2^n = 2^{n+1} - 1$ for every $n \in \mathbb{N}$.

Problem 2.3. Prove that divisibility is a partial order on the natural numbers but not on the integers.