Definition 1. An *equivalence relation* on a set S is a subset $R \subset S \times S$ with the following three properties:

(i) (reflexivity) if $x \in S$ then $(x, x) \in R$;

(ii) (symmetry) if $(x, y) \in R$ then $(y, x) \in R$;

(iii) (transitivity) if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

Definition 2. Let S be a set and R an equivalence relation on S. An *equivalence class* of R is a subset $T \subset S$ such that the following conditions hold:

(i) $R \neq \varnothing$

(ii) $\forall\, x, y \in S,\ \Big((x \in T \wedge y \in T) \implies (x, y) \in R\Big)$

(iii) $\forall\, x, y \in S,\ \Big((x \in T \wedge (x, y) \in R) \implies y \in T\Big)$

The set of all equivalence classes of R on A is denoted A/R and is pronounced A *modulo* R.

The *equivalence class of an element* $x \in S$ is $[x] = \{y \in S \,:\, (x, y) \in R\}$.

Note that the equivalence class of x depends on the equivalence relation R. If we chose a different equivalence relation, the equivalence class of x would change.

Problem 3. Suppose that R is a relation on a set A and B is a subset of A. The restriction of R to B is the relation $R \cap (B \times B)$ on B. If R is an equivalence relation on A, is $R \cap (B \times B)$ an equivalence relation on B?

A) Yes B) No C) Sometimes

Solution. A) □

Problem 4. For any equivalence relation R on any set A, we have $A/R \subset 2^A$.

A) True B) False

Solution. A) □

Problem 5. If R is an equivalence relation on a set S with n elements, what is the smallest number of equivalence classes R could have?

A) 0 B) 1 C) n D) 2^n E) ∞

Solution. If $S = \varnothing$ then A); otherwise B). □

Problem 6. If R is an equivalence relation on a set S with n elements, what is the largest number of equivalence classes R could have?

A) 0 B) 1 C) n D) 2^n E) ∞

Solution. C) □

Theorem 7. *Let R be an equivalence relation on a set S. The equivalence class of an element $x \in S$ is an equivalence class.*

Solution. We prove the first axiom. Since R is reflexive, $(x, x) \in R$, so $x \in [x]$. This proves the first axiom.

We prove the second axiom. Suppose that $y, z \in S$ and $y \in [x]$ and $z \in [x]$. We must show that $(y, z) \in R$. By definition of $[x]$, for y and z to be in $[x]$ means that $(x, y) \in R$ and $(x, z) \in R$. Since R is an equivalence relation, it is symmetric, so we may deduce from $(x, y) \in R$ that $(y, x) \in R$. Finally, since R is an equivalence relation, it is transitive, so we may deduce from $(y, x) \in R$ and $(x, z) \in R$ that $(y, z) \in R$, which is exactly what we wanted. This proves the second axiom.

Now we prove the third axiom. Suppose that $y, z \in S$ and $y \in [x]$ and $(y, z) \in R$. We want to show that $z \in [x]$. By definition of $[x]$, if $y \in [x]$ then $(x, y) \in R$. By transitivity of R we deduce from $(x, y) \in R$ and $(y, z) \in R$ that $(x, z) \in R$. Finally, by definition of $[x]$ we learn from $(x, z) \in R$ that $[x] \in R$. This proves the third axiom and completes the proof. □