Definition 1. Let A and B be sets. A *relation from A to B* is a set of ordered pairs (a, b) such that $a \in A$ and $b \in B$. In other words, a relation from A to B is a subset of $A \times B$.

If A is a set then a *relation on A* means a relation from A to A. We often write aRb to mean $(a, b) \in R$.

Definition 2. Suppose that R is a relation on a set A.

We say that R is ...	if ...
reflexive	$\forall\, x \in A,\ (x, x) \in R$
irreflexive	$\forall\, x \in A,\ (x, x) \notin R$
symmetric	$\forall\, x, y \in A,\ (x, y) \in R \implies (y, x) \in R$
antisymmetric	$\forall\, x, y \in A,\ (x, y) \in R \wedge (y, x) \in R \implies x = y$
transitive	$\forall\, x, y, z \in A,\ (x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R$
total	$\forall\, x, y \in A,\ (x, y) \in R \vee (y, x) \in R$
an *equivalence relation*	if R is reflexive, symmetric, and transitive
a *partial order*	if R is reflexive, antisymmetric, and transitive
a *total order*	if R total, antisymmetric, and transitive

Definition 3. If A is a set we write $\Delta_A = \{(a, a) \in A \times A \ :\ a \in A\}$ and call it the *diagonal*. Sometimes this is also called the *identity* of A and is written id_A.

Suppose that R is a relation from A to B. The following relation is called the *inverse relation* of R:

$$R^{-1} = \{(x, y) \in B \times A \ :\ (y, x) \in R\}$$

If A, B, and C are sets and R is a relation from A to B and S is a relation from B to C then $S \circ R$ is the relation from A to C defined as follows:

$$S \circ R = \{(a, c) \ :\ \exists b \in B, (a, b) \in R \wedge (b, c) \in S\}$$

Problem 4. Characterize reflexive, irreflexive, symmetric, antisymmetric, transitive, and total relations R on A using set operations, R, R^{-1}, Δ_A, $R \circ R$, and $A \times A$. Prove your characterization is correct.

Solution. Reflexive: $\Delta_A \subset R$

Irreflexive: $\Delta_A \cap R = \varnothing$

Symmetric: $R = R^{-1}$

Antisymmetric: $R \cap R^{-1} \subset \Delta_A$

Transitive: $R \circ R \subset R$

Total: $R \cup R^{-1} = A \times A$ $\square$