Definition 1. Let A and B be sets. A *relation from A to B* is a set of ordered pairs (a, b) such that $a \in A$ and $b \in B$. In other words, a relation from A to B is a subset of $A \times B$.

If A is a set then a *relation on A* means a relation from A to A. We often write aRb to mean $(a, b) \in R$.

Definition 2. Suppose that R is a relation on a set A.

We say that R is ...	if ...
reflexive	$\forall\, x \in A,\ (x, x) \in R$
irreflexive	$\forall\, x \in A,\ (x, x) \notin R$
symmetric	$\forall\, x, y \in A,\ (x, y) \in R \implies (y, x) \in R$
antisymmetric	$\forall\, x, y \in A,\ (x, y) \in R \wedge (y, x) \in R \implies x = y$
transitive	$\forall\, x, y, z \in A,\ (x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R$
total	$\forall\, x, y \in A,\ (x, y) \in R \vee (y, x) \in R$
an *equivalence relation*	if R is reflexive, symmetric, and transitive
a *partial order*	if R is reflexive, antisymmetric, and transitive
a *total order*	if R total, antisymmetric, and transitive

Problem 3. The relation $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \ :\ y - x \in \mathbb{N}\}$ is the same as which of the following relations?

A) $|$ B) $=$ C) $\leq$ D) $\geq$

Solution. C) □

Problem 4. Let S be a set and let 2^S be the set of all subsets of S. Let R be the subset relation on 2^S:

$$R = \{(U, V) \in 2^S \times 2^S \ :\ U \subset V\}$$

For each of the following questions, answer A) for Yes and B) for No.

(i) Is R reflexive on 2^S?

(ii) Is R irreflexive on 2^S?

(iii) Is R symmetric on 2^S?

(iv) Is R antisymmetric on 2^S?

(v) Is R transitive on 2^S?

(vi) Is R total on 2^S?

Solution. (i) B)

(ii) B)

(iii) Depends on S: If $S = \varnothing$ it is symmetric, otherwise it is not symmetric.

(iv) A)

(v) A)

(vi) Depends on S: If $|S| \leq 1$ then it is total; otherwise it is not total.

□

Problem 5. Can a relation be both reflexive and irreflexive?

A) Yes B) No

Solution. A) □

Problem 6. Prove that a total relation on a set is reflexive.

Solution. Suppose that R is a total order on a set A. We must prove that for all $x \in A$ we have $(x, x) \in R$. By definition of totality, if $x, y \in A$ then either $(x, y) \in R$ or $(y, x) \in R$. Applying this to $x = y$, we obtain that either $(x, x) \in R$ or $(x, x) \in R$. That is, $(x, x) \in R$, as desired. □

Definition 7. If A is a set we write $\Delta_A = \{(a,a) \in A \times A \,:\, a \in A\}$ and call it the *diagonal*. Sometimes this is also called the *identity* of A and is written id_A.

Suppose that R is a relation from A to B. The following relation is called the *inverse relation* of R:

$$R^{-1} = \{(x,y) \in B \times A \,:\, (y,x) \in R\}$$

If A, B, and C are sets and R is a relation from A to B and S is a relation from B to C then $S \circ R$ is the relation from A to C defined as follows:

$$S \circ R = \{(a,c) \,:\, \exists b \in B, (a,b) \in R \wedge (b,c) \in S\}$$