Problem 1. Which of the following is true for *all* finite sets A and B?

A) $0 \leq |A \cup B| \leq |A|$

B) $|A| \leq |A \cup B| \leq |A| + |B|$

C) $|A| + |B| \leq |A \cup B|$

D) All of the above

Solution. B) ☐

Problem 2. Suppose that A and B are finite sets. State a formula for the size of $|A \cup B|$ in terms of $|A|$, $|B|$, and $|A \cap B|$ and prove it is correct.

Solution. The formula is

$$|A \cup B| = |A| + |B| - |A \cap B| \tag{$*$}$$

We will prove this formula by induction on the size of A.

Let $P(n)$ be the sentence "For all sets A of size n and all finite sets B, Equation ($*$) is true."

If $n = 0$ then $A = \varnothing$ and we have

$$|A \cup B| = |B|.$$

On the other side of the equation, we have

$$|A| + |B| - |A \cap B| = 0 + |B| - 0 = |B|$$

and this agrees with $|B \cup C|$.

Now we proceed to the induction step. Assuming that $P(n)$ holds for a particular natural number n, we prove it also holds for $n + 1$. Pick an element x of A and let $A' = A - \{x\}$. Notice that A' has $n - 1$ elements. We consider two possibilities, depending on whether x is in B:

(i) If $x \notin B$ then the left side of Equation ($*$) is

$$|A \cup B| = |A' \cup B| + 1.$$

The right side is

$$|A| + |B| - |A \cap B| = |A'| + 1 + |B| - |A' \cap B|.$$

(Note that $A \cap B = A' \cap B$ because $x \notin B$.) But the induction hypothesis says that $|A' \cup B| = |A'| + |B| - |A' \cap B|$, so the left and right sides of Equation ($*$) coincide.

(ii) If $x \in B$ then the left side of Equation ($*$) is

$$|A \cup B| = |A' \cup B|.$$

The right side is

$$|A| + |B| - |A \cap B| = |A'| + 1 + |B| - (|A' \cap B| - 1) = |A'| + |B| - |A' \cap B|.$$

By the induction hypothesis, we know that $|A' \cup B| = |A'| + |B| - |A' \cap B|$, we may conclude that $|A \cup B| = |A| + |B| - |A \cap B|$.

This proves the induction step, so by induction we conclude that for every natural number n, the sentence $P(n)$ is true. That is, for all finite sets A and B, we have $|A \cup B| = |A| + |B| - |A \cap B|$. ☐

Solution. Notice that the three sets $A - B$, $B - A$, and $A \cap B$ are pairwise disjoint. Therefore

$$|(A - B) \cup (A \cap B) \cup (B - A)| = |A - B| + |A \cap B| + |B - A|.$$

On the other hand,

$$(A - B) \cup (A \cap B) \cup (B - A) = A \cup B$$

and

$$
\begin{aligned}
|A - B| &= |A| - |A \cap B| \\
|B - A| &= |B| - |B \cap A|.
\end{aligned}
$$

Putting all of this together, we get

$$
\begin{aligned}
|A \cup B| &= |(A - B) \cup (A \cap B) \cup (B - A)| \\
&= |A - B| + |A \cap B| + |B - A| \\
&= |A| - |A \cap B| + |A \cap B| + |B| - |B \cap A| \\
&= |A| + |B| - |A \cap B|,
\end{aligned}
$$

which is exactly what we wanted. $\square$