Problem 1. Prove that the first player can always win the game of Nim if the initial number of stones is $3k$ or $3k-1$ for some natural number k.

Solution. The proof is by induction on k. If $k=0$ then there must be 0 stones (there can't be -1 stones, of course). In this case, the first player wins by definition.

Assume, for the sake of induction, that k is a natural number, $k>0$, and that when there are $3k$ or $3k-1$ stones, the first player can win. We prove that if there are $3(k+1)=3k+3$ or $3(k+1)-1=3k+2$ stones then the first player can also win. Indeed, if there are $3k+3$ stones, the first player takes away 2 stones, and if there are $3k+2$ stones, the first player takes away one stone. Either way, the second player is left with $3k+1$ stones. If the second player takes away 1 stone, the first player will have $3k+1-1=3k$ stones and can win (by induction); if the second player takes away 2 stones, the first player will be left with $3k+1-2=3k-1$ stones and can win (by induction).

By induction, we conclude that the first player wins whenever there is a natural number k such that there are $3k$ or $3k-1$ stones. □

Problem 2. Prove that for every integer n there is a unique choice of an integer k and an $r \in \{0,1,2\}$ such that

$$n = 3k + r.$$

Note that there are two parts of this problem: First, show that k and r exist, and second, show that they are unique.

Solution. The proof is by bidirectional induction on n. First we show that there is at least one integer that can be written as $3q+r$ with $q \in \mathbb{Z}$ and $r \in \{0,1,2\}$. This holds for $n=0$, since we may take $q=r=0$.

The induction step has two parts. First we show that if $n \in \mathbb{Z}$ can be writen as $3q+r$ with $q \in \mathbb{Z}$ and $r \in \{0,1,2\}$ then $n+1$ can also be written as $3q'+r'$ with $q' \in \mathbb{Z}$ and $r' \in \{0,1,2\}$. Assume that $n=3q+r$. If $r=0$ or $r=1$ then we may take $q=q'$ and $r=r'+1$ and then

$$n+1 = 3q+r+1 = 3q'+r',$$

as desried. If $r=2$ then we take $q'=q+1$ and $r'=0$. Then we have

$$n+1 = 3q+r+1 = 3q+3 = 3(q+1)+0 = 3q'+r'$$

as desired.

The second part of the induction step is to show that if $n+1 \in \mathbb{Z}$ can be written as $3q+r$ with $q \in \mathbb{Z}$ and $r \in \{0,1,2\}$ then n can also be writen as $3q'+r'$ with $q' \in \mathbb{Z}$ and $r' \in \{0,1,2\}$. If $r=1$ or $r=2$ then we can take $q=q'$ and $r=r'-1$ and then

$$n = (n+1)-1 = 3q+r-1 = 3q'+r'$$

as desired. If $r=0$ then $r'-1$ is not in $\{0,1,2\}$ so instead we take $q'=q-1$ and $r'=2$. Then we get

$$n = (n+1)-1 = 3q+r-1 = 3q-1 = 3q-3+2 = 3(q-1)+2 = 3q'+r',$$

again as desired. This proves the second part of the induction step.

We conclude by bidirectional induction that every integer n can be written as $3q+r$ for *some* integer q and some $r \in \{0,1,2\}$.

It still remains to show that this representation is unique. Suppose that $n=3q+r$ and $n=3q'+r'$ where q and q' are integers and $r,r' \in \{0,1,2\}$. Then

$$3q+r = 3q'+r'.$$

We can rearrange this to

$$3(q-q') = r'-r,$$

which says that $3|r'-r$. On the other hand, r and r' are both between 0 and 2 so $-2 \le r'-r \le 2$. The only integer in that range that is divisible by 3 is 0, so we must have $r'=r$. This means that $3(q-q')=0$, which means $q'-q=0$. Thus $q=q'$. We conclude that there is at most one way of representing n as $3q+r$ with $q \in \mathbb{Z}$ and $r \in \{0,1,2\}$. □