Problem 1. Suppose I saw a parade of 100 elephants. Every time I saw a pink elephant, the next elephant *and the previous elephant* in the parade were also pink. What is the minimal additional information I would need to deduce that every elephant I saw was pink?

A) There was at least one pink elephant.

B) The first elephant was pink.

C) The last elephant was pink.

D) Both the first and last elephants were pink.

E) Every elephant was pink.

Solution. A) □

Problem 2. Prove that every integer is either even or odd.

Problem 3. Suppose I saw a parade of 100 elephants. Suppose that whenever the n-th elephant was pink, so were the $(n+2)$-th elephant and the $(n-2)$-th elephant. What is the minimal additional information you would need to deduce every elephant I saw was pink?

A) At least one elephant was pink.

B) The first two elephants were pink.

C) Two elephants in a row were pink.

D) An even numbered elephant was pink.

E) One even numbered and one odd numbered elephant were pink.

Solution. E) □

Problem 4. Suppose there was a parade of elephants, one for each natural number. Suppose that whenever all elephants preceding the n-th elephant are pink, the n-th elephant is also pink. What is the minimal additional information you would need to deduce that every elephant in the parade was pink?

A) No additional information.

B) At least one elephant was pink.

C) The first elephant was pink.

D) All the elephants were pink.

Solution. A) □

The principle of strong induction says that a statement of the form $\forall n \in \mathbb{N},\ P(n)$ is equivalent to

$$\forall n \in \mathbb{N}, \Big(\big(\forall m \in \mathbb{N},\ m < n \implies P(m)\big) \implies P(n)\Big)$$

In other words, one may demonstrate the sentence "$P(n)$ holds for all natural numbers n" by showing that

> For every natural number n, if $P(m)$ holds for every natural number $m < n$ then $P(n)$ also holds.

Problem 5. Prove that every natural number other than 1 is divisible by a prime number.

Solution. Let $P(n)$ be the statement "$n = 1$ or n is divisible by a prime number." We prove $P(n)$ for all natural numbers n by strong induction on n.

Suppose that n is a natural number and for all natural numbers $m < n$ we know $P(m)$ is true. We prove that $P(n)$ is true. We separate several cases:

(i) If $n = 0$ then $2|0$ because $0 = 2 \times 0$ so $P(0)$ is true.

(ii) If $n = 1$ then $P(1)$ is obviously true.

(iii) If $n > 1$ and n is prime then $n|n$ so n is divisible by a prime number. Therefore $P(n)$ is true.

(iv) If $n > 1$ and n is not prime then, by definition of primality, there must be an integer m such that $1 < m < n$ and $m|n$. (Note that we couldn't make this conclusion without the knowledge that $n > 1$!) By the induction hypothesis, $P(m)$ is true. Therefore there is a prime p such that $p|m$. Now, $p|m$ and $m|n$ so by Proposition 5.3 of the textbook, $p|n$. Thus n is divisible by a prime number and $P(n)$ is true.

Every natural number falls into one of these cases, so we have proved the induction step. We conclude by strong induction that $P(n)$ is true for all natural numbers n. That is, for all natural numbers n, either $n = 1$ or n is divisible by a prime number. $\square$