**Theorem 1.** *There is no integer that is both even and odd.*

*First proof of Theorem 1.* Suppose, for the sake of contradiction, that $x$ were both even and odd. Then by definition of evenness, $x = 2y$ for some integer $y$. By definition of oddness, $x = 2z + 1$ for some integer $z$. Thus,

$$2y = 2z + 1.$$

This can be rearranged to the equation $1 = 2y - 2z = 2(y - z)$. As $y$ and $z$ are integers, so is $y - z$. By definition of divisibility, this means that 2 divides 1. But we saw earlier that the only integers dividing 1 are 1 and $-1$, so this is a contradiction. Our original assumption that $x$ was both even and odd must have been false. We conclude that no integer $x$ that is both even and odd. □

We will do one more proof using several lemmas:

**Lemma 2.** *If $a$ is an even integer and $b$ is an odd integer then $a - b$ is an odd integer.*

*Proof.* For $a$ to be even means $a = 2c$ for some integer $c$ and for $b$ to be odd menas that $b = 2d + 1$ for some integer $d$. Therefore

$$a - b = 2c - (2d + 1) = 2c - 2d - 1 = 2c - 2d - 2 + 1 = 2(c - d - 1) + 1.$$

As $c - d - 1$ is a sum of integers, it must be an integer, so by the definition of oddness, $a - b$ is odd. □

**Lemma 3.** *If an integer $n$ is odd then $n + 1$ is even.*

*Proof.* If $n$ is odd then by definition we can write $n = 2m + 1$ for some integer $m$. Then $n + 1 = 2m + 2 = 2(m + 1)$. Note that $m + 1$ is the sum of two integers, hence is an integer. Therefore $n + 1$ is even by the definition of evenness. □

**Lemma 4.** *The number 1 is not even.*

*Proof.* We will show that, for every integer $m$, we have $2m \neq 1$. If $m \leq 0$ then $2m \leq 0 < 1$ and if $m \geq 1$ then $2m \geq 2 > 1$. Every integer falls into one of these two cases, so no matter what integer $m$ is, $2m \neq 1$.

This means that 1 is not divisible by 2, so by definition of evenness, 1 is not even. □

*Second proof of Theorem 1.* Suppose, for the sake of contradiction, that it is possible to find an integer that is both even and odd. Let $x$ be such an integer. Then $0 = x - x$ is the difference of an even number ($x$) and an odd number ($x$), hence is odd by Lemma 2. By Lemma 3, this means that 1 is even, in contradiction to Lemma 4. The original assumption, that it is possible to find an integer that is both even and odd, is therefore false. □

**Problem 5.** Which of the two proofs above is better?

**Theorem 6.** *For any integers $x$ and $y$,*

| *If $x$ is ...* | *and $y$ is ...* | *then $x + y$ is ...* |
|---|---|---|
| *even* | *even* | *even* |
| *even* | *odd* | *odd* |
| *odd* | *even* | *odd* |
| *odd* | *odd* | *even* . |

**Problem 7.** Suppose $x$, $y$, $z$, and $w$ are odd numbers. Is $x + y + z + w$ even or odd? Justify your answer with a proof.

A) Even B) Odd C) Depends

*Solution.* A) □

**Problem 8.** Is the sum of 5 odd numbers even or odd? Justify your answer with a proof.

A) Even B) Odd C) Depends

*Solution.* B) □

**Problem 9.** Is the sum of an odd number of odd numbers even or odd? Justify your answer with a proof.

A) Even  B) Odd  C) Depends

*Solution.* B) □

**Problem 10.** Suppose I saw a parade of 100 elephants. Every time I saw a pink elephant, the next elephant in the parade was also pink. What is the smallest possible number of pink elephants I could have seen?

A) 0

B) 1

C) Between 2 and 99

D) 100

*Solution.* A) □

**Problem 11.** Suppose I saw a parade of 100 elephants *and at least one of them was pink*. Every time I saw a pink elephant, the next elephant in the parade was also pink. How many pink elephants did I see?

A) 0

B) 1

C) Between 2 and 99

D) 100

*Solution.* B) □

**Problem 12.** Suppose I saw a parade of 100 elephants *and the first elephant in the parade was pink*. Every time I saw a pink elephant, the next elephant in the parade was also pink. How many pink elephants did I see?

A) 0

B) 1

C) Between 2 and 99

D) 100

*Solution.* D) □

**Problem 13.** Prove that, for every natural number $n$,

$$1 + 3 + 5 + \cdots + (2n + 1) = (n + 1)^2.$$

*Solution.* We are supposed to show that, for every natural number $n$, the formula

$$\sum_{k=0}^{n}(2k + 1) = (n + 1)^2 \tag{$*$}$$

holds. We proceed by induction on $n$.

Base case: $n = 0$. In this case, the left side of Equation ($*$) is $\sum_{k=0}^{0}(2k + 1) = 2 \times 0 + 1 = 1$. The right side is $(0 + 1)^2 = 1$. Both sides equal 1, so the formula holds when $n = 0$.

Induction step: Assuming that Equation ($*$) holds, we prove that

$$\sum_{k=0}^{n+1}(2k + 1) = ((n + 1) + 1)^2. \tag{$\dagger$}$$

We work on the left side:

$$\sum_{k=0}^{n+1}(2k+1) = \sum_{k=0}^{n}(2k+1) + (2(n+1)+1).$$

Using the induction assumption (Equation ($*$)), this is equal to

$$(n+1)^2 + 2(n+1) + 1.$$

On the other hand, the right side of Equation ($\dagger$) expands to

$$((n+1)+1)^2 = (n+1)^2 + 2(n+1) + 1.$$

This agrees with the left side, so we have proved the induction step.

By induction, we conclude that Equation ($*$) holds for all natural numbers $n$. $\square$