Problem 1. Prove that there is a set with an odd number of subsets.

Solution. Let $S = \varnothing$. Then the only subset of S is $\varnothing$ itself, so S has exactly 1 subset. As 1 is an odd number, $\varnothing$ is a set with an odd number of subsets. $\square$

Problem 2. Prove that every integer that is divisible by 10 is also divisible by 5.

Solution. Suppose that x is an integer that is divisible by 10. Then, by the definition of divisibility, there is an integer y such that $x = 10y$. We can rewrite this as $x = 5(2y)$. Now, $2y$ is an integer because both 2 and y are integers, so again by the definition of divisibility, x is divisible by 5.

This holds for any integer that is divisible by 10 so we conclude that every integer that is divisible by 10 is also divisible by 5. $\square$

Problem 3. Prove that not every integer that is divisible by 5 is also divisible by 10.

Solution. We must find an integer that is divisible by 5 but is not divisible by 10. Let us take $x = 5$. As $5 = 5 \times 1$ and both 1 and 5 are integers, the definition of divisibility tells us that $5|5$.

We still have to check that 5 is not divisible by 10. By the definition of divisibility, we must show that there is no integer y such that $10y = 5$. In other words, we must show that, for any integer y, we have $10y \neq 5$. We prove this by noting that any integer must be ≤ 0 or ≥ 1.

1) If $y \leq 0$ then
$$10y \leq 10 \times 0 = 0 < 5$$
so $10y \neq 5$.
2) If $y \geq 1$ then
$$10y \geq 10 \times 1 = 10 > 5$$
so $10y \neq 5$.

As every integer falls into one of these two cases, this shows that there is no integer y such that $10y = 5$.

We conclude that $5|5$ and $10 \nmid 5$. $\square$