Math 2001 Assignment 39

Your name here

Due Wednesday, December 3

Reading 1. Scheinerman, §16 (pp. 85–88)

Definition 2. If $f : T \to U$ is any function and $V \subset U$ is a subset then the *pre-image* of V in T is the set

$$f^{-1}V = \{t \in T \,:\, f(t) \in V\}.$$

Theorem 3. *If $f : A \to B$ is any function then*

$$|A| = \sum_{b \in B} |f^{-1}\{b\}|.$$

Definition 4. If S is a set, let $\binom{S}{k}$ be the set of all subsets of S of size k. Let $(S)_k$ be the set of all k-element lists *without repetition* drawn from S.

Theorem 5. *If S is a finite set of size n and k is an integer such that $0 \le k \le n$ then* $\left|\binom{S}{k}\right| = \frac{n!}{(n-k)! \times k!}$.

Notation 6. Because of Theorem 5, the number $\left|\binom{S}{k}\right|$ only depends on $|S|$ and not on the particular set S we choose. Therefore, we write $\binom{n}{k}$ to mean $\left|\binom{S}{k}\right| = \frac{n!}{(n-k)! \times k!}$ for any finite set S of size n.

Problem 7. Prove the following formula for all integers n and k such that $0 \le k \le n$:

$$\sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!} = 2^n.$$

Use the following steps:

(i) Let S be any set with n elements and let $f : 2^S \to \{0, \dots, n\}$ be the function defined by $f(U) = |U|$. Show that f actually is a function with the indicated domain and codomain. (What do you have to check?)

(ii) Use Theorems 3 and 5 above to prove the desired formula.

Problem 8. (i) How many ways are there to arrange 10 people around a circular table with 10 seats? Two arrangements are considered the same if one is a rotation of the other. (Hint: Let A be the set of arrangements in a line and let B be the set of arrangements in a circle. Find a function $f : A \to B$ and use the theorem.)

(ii) How many ways are there to arrange 5 men and 5 women, alternating between men and women from seat to seat?

Problem 9. Scheinerman, §16, #1

Problem 10. Scheinerman, §17, #1

Problem 11. Scheinerman, §17, #5. (You may use the notation $\binom{n}{k}$ in your answer.)