Math 2001 Assignment 35

Your name here

Due Monday, November 17

Problem 1. Scheinerman, §24, #11

Problem 2. Scheinerman, §24, #14

Problem 3. Scheinerman, §24, #18. Please use the definition from class: A function $f : A \to B$ is a bijection if there is a function $g : B \to A$ such that $g(f(a)) = a$ for every $a \in A$ and $f(g(b)) = b$ for every $b \in B$.

Problem 4. Scheinerman, §24, #19ab

Problem 5. Scheinerman, §24, #23

Problem 6. Let S be a set and k a natural number. In class we introducted the following notation:

(i) S^k is the set of all lists of length k chosen from the elements of S.

(ii) $(S)_k$ is the set of all lists of length k *without repetition* chosen from the elements of S.

Do the following:

(a) Give an example of a set S where S and $(S)_2$ are not disjoint.

Solution. Let $S = \{1, 2, (1, 2)\}$. Then $(1, 2) \in (S)_2 \cap S$. □

(b) Prove that $|S^1| = |S|$ by constructing a bijection.

(c) Prove that $|S^2| = |(S)_2| + |(S)_1|$ by constructing bijections. (Hint: Find a partition of S^2 into two disjoint subsets of sizes $|(S)_2|$ and $|(S)_1|$. This proof was sketched at the end of class; please fill in the details.)

(d) Prove that $|S^3| = |(S)_3|+3|(S)_2|+|(S)_1|$ by constructing bijections. (Hint: Break S^3 into 5 subsets.)

Solution. Let $A = (S)_3 \subset S^3$. Let B be the subset of all triples $(a, a, b) \in S^3$ with $a \neq b$; let C be the subset of all triples $(a, b, a) \in S^3$ with $a \neq b$; let D be the subset of all triples $(b, a, a) \in S^3$ with $a \neq b$; let E be the subset of all triples $(a, a, a) \in S^3$.

The sets A, B, C, D, E are pairwise disjoint. Indeed, everything in A has no repetitions and everything in the other sets has at least one repetition, so A has nothing in common with the others. Everything in E has two repetitions, and B, C, and D all have 1 repetition, so there is nothing in common between E and any of the others. The repetitions in the elements of B, C, and D all occur in different places, so no two of B, C, and D can have anything in common. Therefore

$$|S^3| = |A| + |B| + |C| + |D| + |E|. \tag{$*$}$$

There is a bijection $B \to (S)_2$ by $f(a,a,b) = (a,b)$. The inverse is $g(a,b) = (a,a,b)$; there is a bijection $C \to (S)_2$ by $f(a,b,a) = (a,b)$ with inverse $g(a,b) = (a,b,a)$; there is a bijection $D \to (S)_2$ by $f(b,a,a) = (a,b)$ with inverse $g(a,b) = (b,a,a)$. There is a bijection $E \to (S)_1$ by $f(a,a,a) = (a)$ and $g(a) = (a,a,a)$. We therefore have

$$|A| = |(S)_3| \qquad |B| = |C| = |D| = |(S)_2| \qquad |E| = |(S)_1|.$$

Substituting this into Equation ($*$), we get

$$|S^3| = |(S)_3| + 3|(S)_2| + |(S)_1|.$$

□

(e) Conjecture a formula for $|S^4|$ in terms of $|(S)_4|$, $|(S)_3|$, $|(S)_2|$, and $|(S)_1|$.

Solution. $|S^4| = |(S)_4| + 6|(S)_3| + 7|(S)_2| + |(S)_1|$ □