Math 2001 Assignment 28

Your name here

October 31, 2014

Problem 1. Scheinerman, §15, #6

Problem 2. Suppose that R is an equivalence relation on a set A and B is a subset of A. Prove that $R \cap (B \times B)$ is an equivalence relation on B.

Problem 3. Scheinerman, §15, #17. In this problem you are being asked to give a set S that contains exactly one member of each equivalence class of the similarity relation on triangles.

Problem 4. Let n be an integer. Two integer x and y are said to be congruent modulo n if $n|(x - y)$. For each integer x, let $[x]$ be the equivalence class of x in $\mathbb{Z}$ with respect to congruence modulo n.

If A and B are subsets of $\mathbb{Z}$, define

$$A + B = \{a + b \,:\, a \in A \wedge b \in B\}$$
$$AB = \{ab \,:\, a \in A \wedge b \in B\}.$$

Prove that, for any $x, y \in \mathbb{Z}$, the following formulas hold:

$$[x] + [y] = [x + y]$$
$$[x][y] = [xy].$$

Solution. Here is a proof of the equation $[x] + [y] = [x + y]$.

We prove that $[x] + [y] = [x + y]$. To prove that these two sets are equal, we need to show that every $z \in [x] + [y]$ is also in $[x + y]$ and vice versa.

Suppose first that $z \in [x] + [y]$. We will show that $z \in [x + y]$. By definition of $[x] + [y]$, having $z \in [x] + [y]$ means that $z = a + b$ for some $a \in [x]$ and $b \in [y]$. By definition of $[x]$, we know that $n|a - x$, and by definition of $[y]$, we know that $n|b - y$. By definition of divisibility, there are therefore integers u and v such that $a - x = un$ and $b - y = vn$. Then

$$x + y = (a - un) + (b - vn) = (a + b) - (u + v)n$$

Thus,

$$x + y - z = x + y - (a + b) = -(u + v)n$$

so $n|(x + y - z)$. Thus, $x + y \equiv z \pmod{n}$, so $z \in [x + y]$. This shows that any $z \in [x] + [y]$ is also in $[x + y]$.

Now suppose that $z \in [x+y]$. This means that $n|(x+y-z)$ so there must be an integer w such that $x+y-z = nw$. Then $z = x+(y-nw)$. Note that $x \equiv x \pmod{n}$ and $y \equiv y - nw \pmod{n}$ so $x \in [x]$ and $y - nw \in [y]$. Therefore

$$z = x + (y - nw) \in [x] + [y]$$

This shows that any $z \in [x+y]$ is in $[x]+[y]$.

Putting the two paragraphs together, we conclude that $[x]+[y] = [x+y]$. $\square$