Math 2001-003 Fall 2014
Final Exam Solutions

Wednesday, December 17, 2014

Definition 1. The **union** of two sets X and Y is the set $X \cup Y$ consisting of all objects that are elements of X or of Y. The **intersection** of X and Y is the set $X \cap Y$ consisting of all objects that are elements of both X and Y. The **difference** $X - Y$ of sets X and Y consists of all elements of X that are not elements of Y. The **product** of sets X and Y is the set $X \times Y$ consisting of all ordered pairs (x, y) where $x \in X$ and $y \in Y$.

We say that X is a **subset** of Y, and write $X \subseteq Y$, if every member of X is also a member of Y. Two sets X and Y are said to be disjoint if $X \cap Y = \varnothing$. The **powerset** of a set X is the set of all subsets of X; it is denoted 2^X.

The **cardinality** of a set is the number of elements it contains. Two sets have the same cardinality if there is a bijection from one to the other.

Definition 2. Suppose that X and Y are sets. A **relation** from X to Y is a set R whose elements are ordered pairs (x, y) with $x \in X$ and $y \in Y$. A relation R from X to itself is called a relation on X.

We say that R is ...	if ...
reflexive	$\forall\, x \in A,\ (x, x) \in R$
symmetric	$\forall\, x, y \in A,\ (x, y) \in R \Rightarrow (y, x) \in R$
transitive	$\forall\, x, y, z \in A,\ (x, y) \in R \wedge (y, z) \in R \Rightarrow (x, z) \in R$
an **equivalence relation**	if R is reflexive, symmetric, and transitive .

Definition 3. A relation f from X to Y is said to be a **function** if, for all $x \in X$, there is a unique $y \in Y$ such that $(x, y) \in R$.

The notation $f : X \to Y$ means f is a function from X to Y. If f is a function from X to Y, we write $f(x) = y$ to mean $(x, y) \in f$.

A function $f : X \to Y$ is said to be **injective** if, for all x and x' in X one has $f(x) = f(x')$ only if $x = x'$. We say f is **surjective** if, for all $y \in Y$, there is some $x \in X$ such that $f(x) = y$. We say that f is **bijective** if it is both injective and surjective.

Theorem 4 (Cantor–Schroeder–Bernstein)**.** *Suppose that X and Y are sets and there are injections $f : X \to Y$ and $g : Y \to X$. Then X and Y have the same cardinality.*

Section 1

Problem 1 (6 points)**.** Let $f : A \to B$ be a function. Write a symbolic sentence that means "f is not an injection and f is not a surjection." You may use the following symbols: quantifiers ($\forall, \exists$), logical operators other than negation ($\wedge, \vee, \Rightarrow$), equality and inequality ($=, \neq$). You may **not** use the negation symbol, $\neg$.

Solution.
$$\Big(\exists x \in A,\ \exists y \in A,\ (f(x) = f(y)) \wedge (x \neq y)\Big) \wedge \Big(\exists z \in B,\ \forall x \in A,\ f(x) \neq z\Big)$$

□

Problem 2 (8 points)**.** Rewrite each of the sentences below symbolically using only quantifiers ($\forall, \exists$), logical connectives ($\wedge, \vee, \neg, \Rightarrow$), set operations ($\cup, \cap, \times, -$), and the phrase "is a set". Then determine whether or not the sentence is true. Justify your answers.

(i) (4 points) For all sets S and T, we have $S \times T = T \times S$.

(ii) (4 points) There is a set S such that for all sets T, we have $S \times T = T \times S$.

Problem 3 (10 points)**.** Let $X_0 = 0$, $X_1 = 1$, and define X_n for $n \geq 2$ by recursion:

$$X_n = 5X_{n-1} - 6X_{n-2}$$

Prove that $X_n = 3^n - 2^n$ for all natural numbers n. (Suggestion: Use strong induction.)

Solution. We use strong induction on n. We may assume that $X_m = 3^m - 2^m$ for $m < n$. If $n = 0$, we have $X_0 = 0$ by definition and $3^0 - 2^0 = 1 - 1 = 0$ so the formula holds. If $n = 1$, we have $X_1 = 1$ by definition, and $3^1 - 2^1 = 3 - 2 = 1$, so the formula holds here too.

If $n \geq 2$, we may use the recursive formula $X_n = 5X_{n-1} - 6X_{n-2}$ and substitute $X_{n-1} = 2^{n-1} - 3^{n-1}$ and $X_{n-2} = 2^{n-2} - 3^{n-2}$ to get

$$\begin{aligned}
X_n &= 5X_{n-1} - 6X_{n-2} \\
&= 5(3^{n-1} - 2^{n-1}) - 6(3^{n-2} - 2^{n-2}) \\
&= 5(3 \times 3^{n-2} - 2 \times 2^{n-2}) - 6(3^{n-2} - 2^{n-2}) \\
&= 15 \times 3^{n-2} - 6 \times 3^{n-2} - 10 \times 2^{n-2} + 6 \times 2^{n-2} \\
&= 9 \times 3^{n-2} - 4 \times 2^{n-2} \\
&= 3^n - 2^n
\end{aligned}$$

exactly as desired. □

Problem 4 (8 points)**.** Define sets:

$$A = \{\varnothing, (1,1), 3\}$$
$$B = \{3, 3, \{\varnothing\}, (1,1)\}$$

Compute each of the following sets **and its cardinality**. For each part, your answer should use set builder notation, but may not use any set operations ($\cap, \cup, \times, 2^{(-)}, \Delta$) or the membership symbol ($\in$). (In other words, you should list the elements of the set between curly braces.)

(i) (2 points) $A \cap B$

Solution.

$$A \cap B = \{(1,1), 3\}$$
$$|A \cap B| = 3$$

□

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(ii) (2 points) $A \cup B$

$$A \cup B = \{\varnothing, (1,1), 3, \{\varnothing\}\}$$
$$|A \cup B| = 4$$

(iii) (2 points) $A \times B$

Solution.

$$A \times B = \{(\varnothing, 3), (\varnothing, \{\varnothing\}), (\varnothing, (1,1)), ((1,1), 3), ((1,1), \{\varnothing\}), ((1,1), (1,1)), (3,3), (3, \{\varnothing\}), (3, (1,1))\}$$
$$|A \times B| = 9$$

□

(iv) (2 points) 2^B

$$2^B = \{\varnothing, \{3\}, \{\{\varnothing\}\}, \{(1,1)\}, \{3, \{\varnothing\}\}, \{3, (1,1)\}, \{\{\varnothing\}, (1,1)\}, \{3, \{\varnothing\}, (1,1)\}\}$$
$$|2^B| = 8$$

Problem 5 (10 points)**.** Prove that for any sets A, B, and C the following equation is true:

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

Solution. Suppose that x is an element of $A \times (B \cup C)$. Then $x = (a, y)$ where $a \in A$ and $y \in B \cup C$. This means $y \in B$ or $y \in C$. If $y \in B$ then $(a, y) \in A \times B$ so $x = (a, y) \in (A \times B) \cup (A \times C)$. If $y \in C$ then $x = (a, y) \in A \times C$ so $(a, y) \in (A \times B) \cup (A \times C)$. Either way, $x \in (A \times B) \cup (A \times C)$, so every element of $A \times (B \cup C)$ is also an element of $(A \times B) \cup (A \times C)$.

Conversely, suppose that $x \in (A \times B) \cup (A \times C)$. Then $x \in A \times B$ or $x \in A \times C$. Thus $x = (a, y)$ where $y \in B$ or $y \in C$. Either way, $y \in B \cup C$, so $x = (a, y) \in A \times (B \cup C)$. Thus every element of $(A \times B) \cup (A \times C)$ is an element of $A \times (B \cup C)$. □

Problem 6 (6 points)**.** The following definition of a relation has a blank in it. Match each way of filling in the blank shown on the left with a property on the right that the resulting relation has. Note that the relation between the roman numerals on the left and the letters on the right might not be a bijection!

$$R = \{(x, y) \ : \ x^2 = y^2 \ \wedge \ \_\_\_\_\_ \ \}$$

(i) $x \in \mathbb{Z} \wedge y \in \mathbb{Z}$

(ii) $x \in \mathbb{N} \wedge y \in \mathbb{N}$

(iii) $x \in \mathbb{Z} \wedge y \in \mathbb{N}$

(iv) $x \in \mathbb{N} \wedge y \in \mathbb{Z}$

(A) not a function

(B) a function that is injective but not surjective

(C) a function that is surjective but not injective

(D) a bijection

Now describe this relation between roman numerals and letters as a set of ordered pairs.

Solution.

$$\{(i, A), (ii, D), (iii, A), (iv, B)\}$$

□

Problem 7 (5 points)**.** Find an equivalence relation on $\{1, 2, 3, 4, 5, 6\}$ whose equivalence classes are $\{3, 5\}$, $\{1, 6\}$, $\{2\}$, and $\{4\}$.

Solution.

$$R = \{(1,1), (1,6), (6,1), (6,6), (3,3), (3,5), (4,4), (5,3), (5,5), (2,2)\}$$

□

Section 2

Problem 8 (10 points)**.** A **semigroup** is a pair (G, m) where

(i) G is a set;

(ii) $m : G \times G \to G$ is a function;

(iii) for all $a, b, c \in G$ we have $m(a, m(b, c)) = m(m(a, b), c)$.

Fix a set S and let G be the set of all functions from S to itself. For any $a, b \in G$, define $m(a, b) = a \circ b$. Prove that (G, m) is a semigroup.

Solution. By definition, G is a set. Since b is a function with codomain S and a is a function with domain S, the relation $a \circ b$ is a function. The domain of $a \circ b$ is the same as the domain of b, namely S, and the codomain of $a \circ b$ is the same as the codomain of a, namely S. Therefore $a \circ b$ is a function from S to itself, so it is in G.

Now we check the third property. Suppose that $a, b, c \in G$. Then $m(a, m(b, c)) = a \circ (b \circ c)$ and $m(m(a, b), c) = (a \circ b) \circ c$. These are both functions, so in order to show they are identical, we need to show that

$$a \circ (b \circ c)(x) = (a \circ b) \circ c(x)$$

for all $x \in S$. Suppose that $x \in S$. Then

$$a \circ (b \circ c)(x) = a(b \circ c(x)) = a(b(c(x)))(a \circ b) \circ c(x) = a \circ b(c(x)) = a(b(c(x))).$$

As this holds for all $x \in S$, we conclude that $m(a, m(b, c)) = a \circ (b \circ c) = (a \circ b) \circ c = m(m(a, b), c)$, as desired. □

Problem 9 (10 points)**.** Prove that every non-empty finite set has the same number of subsets of even size as it has subsets of odd size.

Solution. Consider the formula $(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. Substitute $x = -1$. As long as $n > 0$, we get 0 on the left side, so we obtain $0 = \sum_{k=0}^{n} \binom{n}{k}(-1)^k$. Move the negative terms to the left side to get

$$\sum_{\substack{0 \le k \le n \\ k \text{ odd}}} \binom{n}{k} = \sum_{\substack{0 \le k \le n \\ k \text{ even}}} \binom{n}{k}.$$

The sum on the left side is the number of odd-sized subsets and the sum on the right is the number of even-sized subsets. □

Solution. Let S be a non-empty set. Let A be the set of even-sized subsets of S and let B be the set of odd-sized subsets of S. Since $S \neq \varnothing$, we can choose some $x \in S$. Construct a function $f : A \to B$ as follows: If $T \in A$ and $x \in T$ then $f(T) = T - \{x\}$. If $T \in A$ and $x \notin T$ then $f(T) = T \cup \{x\}$. Notice that $|f(T)| - |T| = \pm 1$ so that if $|T|$ is even then $|f(T)|$ is odd. The same formula defines a function $g : B \to A$.

We check that f and g are inverses. If $x \notin T$ and $T \in A$ then $g(f(T)) = g(T \cup \{x\}) = (T \cup \{x\}) - \{x\} = T$. If $x \in T$ then $g(f(T)) = g(T - \{x\}) = (T - \{x\}) \cup \{x\} = T$. Since f and g are defined the same way, we also get $f(g(T)) = T$ for all $T \in B$. Thus f and g are inverse functions and $|A| = |B|$, as desired. □

Problem 10 (10 points)**.** Compute the number of ways to partition a set of 24 elements into three sets of size 4, two sets of size 5, and one set of size 2. Express your answer using only multiplication, division, exponentiation, and the factorial. Justify your answer.

Solution. Let S be a set with 24 elements, and let X be the set of all lists

$$(A_1, A_2, A_3, B_1, B_2, C_1)$$

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where $|A_i| = 4$ for all i, $|B_i| = 2$ for all i, and $|C_i| = 1$ for all i. We can calculate the size of X: There are $\binom{24}{4}$ choices for A_1, then $\binom{20}{4}$ choices for A_2, then $\binom{16}{4}$ choices for A_3, then $\binom{12}{5}$ choices for B_1, then $\binom{7}{5}$ choices for B_2, and then $\binom{2}{2}$ choices for C_1. This gives

$$\begin{aligned} |T| &= \binom{24}{4}\binom{20}{4}\binom{16}{4}\binom{12}{5}\binom{7}{5}\binom{2}{2} \\ &= \frac{24!}{20! \times 4!} \times \frac{20!}{16! \times 4!} \times \frac{16!}{12! \times 4!} \times \frac{12!}{7! \times 5!} \times \frac{7!}{5! \times 2!} \times \frac{2!}{2! \times 0!} \\ &= \frac{24!}{4!^3 \times 5!^2 \times 2!}. \end{aligned}$$

Now let Y be the set of all partitions of S as required by the problem. There is a function

$$f : X \to Y f((A_1, A_2, A_3, B_1, B_2, C_1)) = \{A_1, A_2, A_3, B_1, B_2, C_1\}.$$

For each partition $P \in Y$ there are $3! \times 2! \times 1!$ lists L such that $f(L) = P$, corresponding to the ways of rearranging the As, the Bs, and the Cs in any given list. Therefore $|f^{-1}\{P\}| = 3! \times 2! = 12$ for all $P \in Y$. We conclude that

$$\frac{24!}{4!^3 \times 5!^2 \times 2!} = |X| = \sum_{P \in Y} |f^{-1}\{P\}| = \sum_{P \in Y} 3! \times 2! = 3! \times 2! \times |Y|$$

so

$$|Y| = \frac{24!}{4!^3 \times 5!^2 \times 3! \times 2!^2}.$$

□

Problem 11 (10 points)**.** Prove that the cardinality of the set of natural numbers, $\mathbb{N}$, is the same as the cardinality of the set of integers, $\mathbb{Z}$.

Solution. We construct injections in both directions and use the Cantor–Schroeder–Bernstein theorem. The function $f(x) = x$ defines an injection from $\mathbb{N}$ to $\mathbb{Z}$.

To get an injection in the other direction, define $f(n) = 2n$ if $n \geq 0$ and $f(n) = -2n - 1$ if $n < 0$. We check that $f : \mathbb{Z} \to \mathbb{N}$. If $n \in \mathbb{Z}$ then either $n \geq 0$ or $n \leq -1$. If $n \geq 0$ then $2n$ is a non-negative integer, hence is in $\mathbb{N}$. If $n \leq -1$ then $-2n - 1 \geq -2(-1) - 1 = 1$ is a positive integer, hence is a natural number. Thus $f(n) \in \mathbb{N}$ for all $n \in \mathbb{Z}$.

We still have to check f is injective. Suppose $m, n \in \mathbb{Z}$ and $f(n) = f(m)$. □

Problem 12 (10 points)**.** Prove that, for every integer n, there is an integer m such that $n = 3m$ or $n = 3m + 1$ or $n = 3m + 2$.

Solution. First we prove this for natural numbers using induction. If $n = 0$ then we can write $n = 3 \times 0$ so the statement is true for $n = 0$. Now we assume that there is an integer m such that $n = 3m$ or $n = 3m + 1$ or $n = 3m + 2$ and prove that there is an integer ℓ such that $n + 1 = 3\ell$ or $n + 1 = 3\ell + 1$ or $n + 1 = 3\ell + 2$. We consider three cases: If $n = 3m$ then we can take $\ell = m$ and we have $n + 1 = 3m + 1$; if $n = 3m + 1$, we can take $m = \ell$ and $n + 1 = 3m + 2$; if $n = 3m + 2$ then we have $n + 1 = 3m + 3 = 3(m + 1)$ so we can take $\ell = m + 1$. This proves the inductive step and completes the proof for $n \in \mathbb{N}$.

We still need to consider the possibility that $n < 0$. For that case, notice that $-n \geq 0$, so by what we have already proved, there is an integer m such that $-n = 3m$ or $-n = 3m + 1$ or $-n = 3m + 2$. In the first case, we get $n = 3(-n)$ which is the form we want; in the second case, we get $n = -3m - 1 = 3(-m - 1) + 2$, which gives what we want; finally, in the third case we have $n = -3m - 2 = 3(-m - 1) + 1$, which takes care of the final case and completes the proof for negative n. □