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# Math 2001-003 Fall 2014
# Midterm Exam 2 Solutions

Thursday, November 6, 2014

**Definition 1.** An integer $m$ is said to **divide** an integer $n$ if there is an integer $c$ such that $n = cm$.

Two integers $a$ and $b$ are said to be **congruent** modulo an integer $n$ if $n|(b-a)$. We write $a \equiv b \pmod{n}$ in this case.

**Definition 2.** The number of elements of a set $X$ is denoted $|X|$ and is known as the **cardinality** or **size** of $X$.

The **union** of two sets $X$ and $Y$ is the set $X \cup Y$ consisting of all objects that are elements of $X$ or of $Y$. The **intersection** of $X$ and $Y$ is the set $X \cap Y$ consisting of all objects that are elements of both $X$ and $Y$.

The **powerset** of a set $X$ is the set of all subsets of $X$; it is denoted $2^X$.

The **difference** $X - Y$ of sets $X$ and $Y$ consists of all elements of $X$ that are not elements of $Y$. The **symmetric difference** $X \,\Delta\, Y$ of $X$ and $Y$ is $(X - Y) \cup (Y - X) = (X \cup Y) - (X \cap Y)$.

We say that $X$ is a **subset** of $Y$, and write $X \subseteq Y$, if every member of $X$ is also a member of $Y$. Two sets $X$ and $Y$ are said to be disjoint if $X \cap Y = \varnothing$.

**Definition 3.** Suppose that $X$ and $Y$ are sets. A **relation** from $X$ to $Y$ is a set $R$ whose elements are ordered pairs $(x, y)$ with $x \in X$ and $y \in Y$.

| We say that $R$ is ... | if ... |
|---|---|
| **reflexive** | $\forall\, x \in A,\ (x, x) \in R$ |
| **irreflexive** | $\forall\, x \in A,\ (x, x) \notin R$ |
| **symmetric** | $\forall\, x, y \in A,\ (x, y) \in R \implies (y, x) \in R$ |
| **antisymmetric** | $\forall\, x, y \in A,\ (x, y) \in R \wedge (y, x) \in R \implies x = y$ |
| **transitive** | $\forall\, x, y, z \in A,\ (x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R$ |
| **total** | $\forall\, x, y \in A,\ (x, y) \in R \vee (y, x) \in R$ |
| an **equivalence relation** | if $R$ is reflexive, symmetric, and transitive |
| a **partial order** | if $R$ is reflexive, antisymmetric, and transitive |
| a **total order** | if $R$ total, antisymmetric, and transitive |

If $R$ is a total order on a set $A$ then an element $a \in A$ is said to be **minimal** or the **least element** of $A$ if, for all $b \in A$, we have $(a, b) \in R$.

**Theorem 4.** *If $A$ and $B$ are finite sets then*

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$|A - B| = |A| - |A \cap B|.$$

**Problem 1.** (6 points) Identify an equivalence relation $R$ on a set $A$ with the following equivalence classes:

$$\{\varnothing, 1\}$$
$$\{\{\varnothing\}, 2\}$$

No justification required.

*Solution.* Let $A = \{\varnothing, \{\varnothing\}, 1, 2\}$ and define

$$R = \{(\varnothing, \varnothing), (\varnothing, 1), (1, \varnothing), (1, 1), (\{\varnothing\}, \{\varnothing\}), (\{\varnothing\}, 2), (2, \{\varnothing\}), (2, 2)\}$$

□

**Problem 2.** (6 points) Construct a set $A$ and a relation $R$ on $A$ that is reflexive, irreflexive, symmetric, antisymmetric, transitive, and total. Justify your answer. (It is possible to give a complete justification in one sentence.)

*Solution.* Let $A = R = \varnothing$. All of the desired properties are universally quantified over elements of $A$, so they hold vacuously. □

**Problem 3.** (10 points) Suppose that $a \equiv b \pmod{n}$ and that $m|n$. Prove that $a \equiv b \pmod{m}$.

*Solution.* Suppose that $a \equiv b \pmod{n}$ and that $m|n$. By definition of congruence modulo $n$, this means that $n|(b-a)$, which means that there is an integer $k$ such that $kn = b - a$, by definition of divisibility. Also by definition of divisibility, there is an integer $\ell$ such that $n = \ell m$. Substituting this into $kn = b - a$, we get $k\ell m = b - a$. But $k\ell$ is a product of integers, hence is an integer, so $m|(b-a)$, by definition of divisibility. By definition of congruence modulo $m$, we conclude that $a \equiv b \pmod{m}$. □

**Problem 4.** (8 points) Let $A = \{1, 2, 3, 4\}$ and let $R$ be the following relation on $A$:

$$R = \{(1, 1), (1, 3), (3, 1), (3, 4)\}$$

For each of the properties named below, indicate the **smallest** set $R'$ such that $R \cup R'$ is a relation on $A$ with the named property. No justification required.

(i) reflexive

*Solution.* $R' = \{(2, 2), (3, 3), (4, 4)\}$ □

(ii) symmetric

*Solution.* $R' = \{(4, 3)\}$ □

(iii) transitive

*Solution.* $R' = \{(1, 4), (3, 3)\}$ □

(iv) equivalence relation

*Solution.* $R' = \{(1, 4), (2, 2), (3, 3), (4, 1), (4, 3), (4, 4)\}$ □

**Problem 5.** (14 points)

(i) (7 points) Prove that if $x$ and $y$ are real numbers such that $xy = 0$ then $x = 0$ or $y = 0$.

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*Solution.* Suppose, for the sake of contradiction, that there are real numbers $x$ and $y$ such that $xy = 0$ but $x \neq 0$ and $y \neq 0$. Since $x \neq 0$, we may divide by $x$ on both sides of $xy = 0$ to get $y = x^{-1} \times 0 = 0$. This contradicts the assumption $y \neq 0$. The original assumption, that there are real numbers $x$ and $y$ such that $xy = 0$ but $x \neq 0$ and $y \neq 0$ must therefore have been false. We conclude that, for all real numbers $x$ and $y$, if $xy = 0$ then $x = 0$ or $y = 0$. $\square$

(ii) (7 points) Disprove that if $x$, $y$, and $n$ are integers such that $xy \equiv 0 \pmod{n}$ then $x \equiv 0 \pmod{n}$ or $y \equiv 0 \pmod{n}$.

*Solution.* Let $n = 6$ and let $x = 2$ and let $y = 3$. Then $x \not\equiv 0 \pmod{6}$ and $y \not\equiv 0 \pmod{6}$ since 6 does not divide 2 or 3. On the other hand,

$$2 \times 3 = 6 \equiv 0 \pmod{6}.$$

Therefore $n = 6$, $x = 2$, and $y = 3$ are a counterexample. $\square$

**Problem 6.** (8 points) Prove that for any finite sets $A$ and $B$,

$$|A \,\Delta\, B| = |A| + |B| - 2|A \cap B|.$$

*Solution.* By definition,

$$|A \,\Delta\, B| = |(A \cup B) - (A \cap B)|.$$

Note that $A \cap B \subset A \cup B$. Indeed, if $x \in A \cap B$ then $x \in A$ and $x \in B$, so of course $x \in A$ or $x \in B$, which is what it means for $x$ to be in $A \cup B$. Therefore, $(A \cup B) \cap (A \cap B) = (A \cap B)$ so by Theorem 4,

$$|(A \cup B) - (A \cap B)| = |A \cup B| - |(A \cup B) \cap (A \cap B)| = |A \cup B| - |A \cap B|.$$

On the other hand, also by Theorem 4, we know that

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Putting all of this together, we get

$$\begin{aligned} |A \,\Delta\, B| &= |(A \cup B) - (A \cap B)| \\ &= |A \cup B| - |A \cap B| \\ &= |A| + |B| - |A \cap B| - |A \cap B| \\ &= |A| + |B| - 2|A \cap B|, \end{aligned}$$

exactly as desired. $\square$

**Problem 7.** (12 points) Prove that $n! < n^n$ for every integer $n > 1$. You may use the following facts about real numbers $p$, $q$, and $a$ without proof:

(i) If $0 \leq p < q$ then $p^n < q^n$ for all $n \geq 1$.

(ii) If $p < q$ and $a > 0$ then $ap < aq$.

*Solution.* The proof is by induction on $n$, with the base case $n = 2$. When $n = 2$, we have $2! = 2$ and $2^2 = 4$ so $2! = 2 < 4 = 2^2$, which gives the base case.

Now we proceed by induction. Assume that $n \in \mathbb{N}$ and $n \geq 2$ and $n! < n^n$. We want to prove that $(n+1)! < (n+1)^{n+1}$. By definition of the factorial, we have $(n+1)! = (n+1) \times n!$. On the other hand, the induction hypothesis says that $n! < n^n$. As $0 < n < n+1$, Fact (i) says that $n^n < (n+1)^n$. Fact (ii) then implies that $(n+1) \times n^n < (n+1) \times (n+1)^n$ since $n + 1 > 0$. Putting these together, we get

$$(n+1)! = (n+1) \times n! < (n+1) \times n^n < (n+1) \times (n+1)^n = (n+1)^{n+1}$$

which is exactly what we wanted. $\square$

**Problem 8.** (20 points)

(i) (14 points) Suppose that $S$ is a non-empty finite set and $\prec$ is a total order on $S$. Prove that $S$ has a least element with respect to $\prec$. (Hint: Use induction. You may use without proof that the restriction of a total order to a subset is a total order.)

*Solution.* The proof is by induction on the size of $S$. The base case is $|S| = 1$. In that case, $S$ has just one element—call it $x$. We check that $x$ is the least element of $S$. Since $\prec$ is a total order, it is reflexive, so $x \prec x$. Since $x$ is the only element of $S$, this means that $\forall\, y \in S\,, x \prec y$, as desired.

For the sake of induction, assume that a total order on a set with $n$ elements has a least element. We prove that a total order on a set with $n+1$ elements also has a least element. Let $S$ be a set with $n+1$ elements. Since $n$ is a natural number, $|S| > 0$, so we may choose an element $x \in S$. Then $S - \{x\}$ has $n$ elements. Moreover, the restriction of $\prec$ to $S - \{x\}$ is a total order so there is an element—call it $y$—of $S - \{x\}$ that is least in $S - \{x\}$ with respect to $\prec$.

Now there are two possibilities. Since $\prec$ is a total order, either $y \prec x$ or $x \prec y$.

(i) If $y \prec x$ then $y$ is the least element of $S$. We verify this: Suppose that $z$ is an element of $S$. Either $z = x$ or $z \in S - \{x\}$. If $z = x$ then $y \prec z$ by assumption. If $z \in S - \{x\}$ then $y \prec z$ by the assumption that $y$ was a minimal element of $S - \{x\}$.

(ii) If $x \prec y$ then $x$ is the least element of $S$. We verify this: Suppose that $z$ is an element of $S$. Either $z = x$ or $z \in S - \{x\}$. If $z = x$ then $x \prec z$ because $\prec$ is reflexive. If $z \in S - \{x\}$ then $x \prec y \prec z$ ($x \prec y$ by assumption and $y \prec z$ because $y$ is the minimal element of $S - \{x\}$ and $z \in S - \{x\}$). But $\prec$ is transitive, so $x \prec z$. Thus $x$ is minimal.

In either case there is a minimal element of $S$ with respect to $\prec$. This completes the induction step and the proof. $\square$

(ii) (6 points) Show that a total order on an infinite set does not necessarily have a least element.

*Solution.* The set $\mathbb{Z}$ with the relation $\leq$ is an example of a total order on an infinite set without a minimal element. $\square$