Math 2001-003 Fall 2014
Midterm Exam 1 Solutions

Tuesday, September 23, 2014

Problem 1. (3 points) Use a truth table to demonstrate that $\neg X \vee Y$ and $X \implies Y$ are logically equivalent.

Solution.

X	Y	$\neg X$	$\neg X \vee Y$	$X \implies Y$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The truth values in last two columns are identical, so $\neg X \vee Y$ and $X \implies Y$ are logically equivalent. $\square$

Problem 2. (2 points) Which of the following are logical opposites of $X \implies Y$? No justification required. (Scoring: 1/2 point for each correct answer circled, 1/2 point for each incorrect answer not circled.)

(A) $Y \implies X$ (B) $\neg Y \implies \neg X$ (C) $X \wedge \neg Y$ (D) $X \vee \neg Y$

Solution. (C) $\square$

Problem 3. (1 point) The empty set is an element of every set. No justification required.

(A) True (B) False

Solution. False $\square$

Problem 4. (2 points) Write down the elements of the power set of $\{\{\varnothing\}, 1\}$.

Solution. $\varnothing, \{\{\varnothing\}\}, \{1\}, \{\{\varnothing\}, 1\}$ $\square$

Problem 5. (3 points) Compute the following sum:

$$\sum_{n=0}^{100} \prod_{k=n}^{2} k$$

Your answer should be a single number. Justification is not required, but partial credit will be given for incorrect answers that are partially justified.

Solution.

$$\begin{aligned}\sum_{n=0}^{100} \prod_{k=n}^{2} k &= \sum_{n=0}^{2} \prod_{k=n}^{2} k + \sum_{n=3}^{100} 1 \\ &= 0 \times 1 \times 2 + 1 \times 2 + 2 + 98 \\ &= 102\end{aligned}$$

$\square$

Solution. Here is the same solution, written another way:

$$\sum_{n=0}^{100} \prod_{k=n}^{2} k = \prod_{k=0}^{2} k + \prod_{k=1}^{2} k + \prod_{k=2}^{2} k + \prod_{k=3}^{2} k + \cdots + \prod_{k=100}^{2} k$$
$$= 0 \times 1 \times 2 + 1 \times 2 + 2 + 1 + \cdots + 1$$

Observe that there number 1 is added 98 times (once for every k between 3 and 100, inclusive) yielding a total of

$$0 + 2 + 2 + 98 = 102.$$

□

Problem 6. (5 points) A function f from a real vector space V to another real vector space is said to be *linear* if the following sentence is true:

$$\Big(\forall\, \lambda \in \mathbb{R},\ \forall\, x \in V,\ f(\lambda x) = \lambda f(x)\Big) \wedge \Big(\forall\, x \in V,\ \forall\, y \in V,\ f(x+y) = f(x) + f(y)\Big)$$

Write a symbolic sentence meaning "f is not linear" using only quantifiers ($\forall$, $\exists$), logical connectives other than negation ($\wedge$, $\vee$), set membership ($\in$), equality and inequality ($=$, $\neq$), variables, and the objects $\mathbb{R}$, V, and f. Note: You are not allowed to use negation ($\neg$)! You do not need to know what a function or a vector space is to solve this problem.

Solution.

$$\Big(\exists\, \lambda \in \mathbb{R},\ \exists\, x \in V,\ f(\lambda x) \neq \lambda f(x)\Big) \vee \Big(\exists\, x \in V,\ \exists\, y \in V,\ f(x+y) \neq f(x) + f(y)\Big)$$

□

Problem 7. (5 points) Which integers are divisible by zero? Justify your answer.

Solution. An integer n is divisible by zero if and only if there is an integer c satisfying the equation $n = c \times 0$. But no matter what c is, $c \times 0$ will always be 0, so the only way this equation can have a solution is if $n = 0$. Therefore no integer other than zero is divisible by zero.

Of course, if $n = 0$ then we can choose c to be any integer to make $n = c \times 0$ valid. This shows that 0 is divisible by zero.

We conclude that zero is divisible by zero and that no other integer is divisible by zero. □

Problem 8. (3 points) Which of the following are true for all sets A and all objects x? Circle those that are. (Scoring: 1 point for each correct answer circled; 1 point for each incorrect answer not circled.)

(A) $\{x\} \in A \iff \{\{x\}\} \subseteq 2^A$

(B) $x \subseteq A \implies \{x\} \subseteq 2^A$

(C) $\{x\} \in A \implies x \subseteq A$

Solution. (B) □

Problem 9. (5 points) Suppose that S is a set and $P(x)$ is a sentence that depends on an element x of S. Assume that the sentence $\forall\, x \in S,\ P(x)$ is true. Must the sentence $\exists\, x \in S,\ P(x)$ also be true? Justify your answer.

Solution. No. If S is empty then no matter what $P(x)$ is, the sentence $\forall x \in S$, $P(x)$ is true but $\exists x \in S$, $P(x)$ is false. □

Problem 10. (5 points) Give precise conditions on finite sets X and Y under which the formula $|X - Y| = |X| - |Y|$ is correct. (Your answer should describe a relationship between X and Y such that, if X and Y have that relationship then $|X - Y| = |X| - |Y|$, and if X and Y do not have that relationship then $|X - Y| \neq |X| - |Y|$. The relationship you describe may not make reference to the cardinalities of any sets.)

Solution. The formula $|X - Y| = |X| - |Y|$ holds if and only if $Y \subseteq X$. $\square$

Problem 11. (i) (3 points) Let $X = \{0, 1, 2\}$ and let $Y = \{3, 4\}$. Indicate which of the following sets are relations from X to Y by circling those that are:

$\varnothing$	$X \times Y$
$\{(0,4),(0,3)\}$	$\{(0,0),(1,4)\}$
$X \cup Y$	$X \cap Y$

(Scoring: 1/2 point for each correct answer circled; 1/2 point for each incorrect answer not circled.)

Solution. $\varnothing$, $X \times Y$, $\{(0,4),(0,3)\}$, and $X \cap Y$ are all relations from X to Y. (Note that $X \cap Y = \varnothing$ in this case; for most sets X and Y the set $X \cap Y$ is not a relation from X to Y.) The sets $X \cup Y$ and $\{(0,0),(1,4)\}$ are not relations from X to Y. $\square$

(ii) (5 points) Suppose that X is a set with a elements and Y is a set with b elements. How many different relations are there from X to Y? Justify your answer briefly.

Solution. Observe that a relation from X to Y is the same thing as a subset of $X \times Y$. Therefore the set of all relations from X to Y coincides with $2^{X \times Y}$. This has $2^{|X \times Y|} = 2^{|X| \times |Y|} = 2^{ab}$ elements, so there are 2^{ab} relations from X to Y. $\square$