MATH 2300-004: Limit Review

Limits of Rational Functions at Infinity (Section 2.3)
For sequences, we in particular care about the behavior as $n \to \infty$. So for rational functions (one polynomial dividing another), we have the following rules of thumb:
First, compare the degrees of the top and bottom (i.e., the highest power of n in each).
Then:

If the degrees are equal, then the limit exists and is equal to the ratio of the leading coefficients (the coefficients of the highest power terms. For example,

$$\lim_{n\to\infty} \frac{2n^5 - n}{n^2 + n^5 - 4} = \frac{2}{1} = 2$$

If the degree of the bottom is bigger than the degree of the top, then the limit exists and is equal to zero. For example,

$$\lim_{n\to\infty} \frac{n^8 + 1}{n^9 - 1} = 0$$

If the degree of the top is bigger than the degree of the bottom, then the limit does not exist, and in fact goes to $\pm\infty$. For example,

$$\lim_{n\to\infty} \frac{n^3 + 4n + 1}{n^2 - 2n - 2} = +\infty$$

In the case of sequences, we would say the sequence diverges.

L'Hopital's Rule (Section 4.4)
L'Hopital's Rule is a technique from Calc 1 that uses derivatives to find the limits of indeterminant forms. There are several indeterminant forms, and different techniques for dealing with them.

Indeterminant Form: $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

This is the simplest case: just take the derivative of the top and bottom (separately! without quotient rule!) and check the value of the new limit. For example:

$$\lim_{n\to\infty} \frac{n}{e^{2n}} = \lim_{n\to\infty} \frac{1}{2e^{2n}} = 0$$

Indeterminant Form: $0 \cdot \infty$.

Here, our goal is to rewrite the multiplied functions as a fraction, so that $f(x)g(x)$ becomes $\frac{g(x)}{\frac{1}{f(x)}}$. This transforms the indeterminant form from $0 \cdot \infty$ to $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and we can then use L'Hopital's Rule directly. For example:

$$\lim_{n\to\infty} n^2 \sin(\frac{1}{n}) = \lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\cos(\frac{1}{n})\frac{-1}{n^2}}{\frac{-2}{n^3}} = \lim_{n\to\infty} \frac{1}{2} n \cos(\frac{1}{n}) = \infty$$

Usually, we pick the simpler function to move to the denominator, as it leads to simpler derivatives.

Indeterminant Form: $0^0, \infty^0, 1^\infty$. This one is most complicated. To start, we set y equal to our limit, then take the natural log of both sides. This brings the exponent out in front of the log in the limit, and transforms the indeterminant form into $0 \cdot \infty$, which we can then solve using techniques above. For example:

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = 1^\infty$$

$$y = \lim_{n\to\infty}(1+\frac{1}{n})^n$$

$$\ln(y) = \lim_{n\to\infty} n\ln(1+\frac{1}{n}) = 0 \cdot \infty$$

$$\lim_{n\to\infty} n\ln(1+\frac{1}{n}) = \lim_{n\to\infty}\frac{\ln(1+\frac{1}{n})}{\frac{1}{n}} = \frac{0}{0}$$

$$\lim_{n\to\infty}\frac{\ln(1+\frac{1}{n})}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\frac{1}{1+\frac{1}{n}}\frac{-1}{n^2}}{\frac{-1}{n^2}} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1$$

$$\ln(y) = 1 \to y = e$$

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e$$

It is important to note that once we get a value for the limit, this value is equal to $\ln(y)$, whereas we want the value of y (our limit); thus we use the value we get for the limit as a power of e to get our final answer.

There are other techniques for limits scattered through sections 2.1-2.3 in the book. I recommend you skim through those sections to refresh yourself before the exam. In addition, we can also use the Squeeze Theorem to evaluate some of these limits. Limits you should know:

$$\lim_{n\to\infty}\ln(n) = \infty$$

$$\lim_{n\to\infty}\tan^{-1}(n) = \frac{\pi}{2}$$

$$\lim_{n\to\infty}e^n = \infty$$

$$\lim_{n\to\infty}\sin(n) dne$$

$$\lim_{n\to\infty}\cos(n) dne$$

$$\lim_{n\to\infty}\frac{1}{n^k} = 0 \text{ for any } k > 0$$