First Order Differential Equations

A differential equation is an equation involving one or more derivatives of an unknown function. A function $y(x)$ is a solution of a differential equation if the equation is satisfied when y and its derivatives are substituted into the equation.

■ First Order Differential Equations

A differential equation is called first order if the highest order of any derivative appearing in the equation is the first, i.e. no second or higher order derivatives appear.

We will only be concerned with linear first order differential equations, which have the general form:

$$\frac{dy}{dx} + p(x)y = q(x)$$

▶ **Example 1** Some examples of first order linear differential equations are:

$$\begin{aligned}\frac{dy}{dx} + x^2y &= \sin x \\ y' &= 4y + 1 \\ \frac{ds}{dt} - e^x y &= 0\end{aligned}$$

In each case, what are $p(x)$ and $q(x)$?

Solution

In the first case the differential equation is already in general form with $p(x) = x^2$ and $q(x) = \sin x$.

In the second case the general form is

$$y' - 4y = 1$$

and so $p(x) = -4$ and $q(x) = -1$.

In the last case differential equation is already in general form with $p(x) = -e6x$ and $q(x) = 0$.

■ METHOD OF INTEGRATING FACTORS

We wish to solve the differential equation

$$\frac{dy}{dx} + p(x)y = q(x)$$

We make the following observation. If we define a function $\mu(x)$ by

$$\mu = e^{\int p(x)\,dx}$$

then we have by the product rule and the fundamental theorem of calculus:

$$\frac{d}{dx}(\mu y) = \mu\frac{dy}{dx} + \frac{d\mu}{dx}y = \mu\frac{dy}{dx} + \left(\mu\frac{d}{dx}\int p(x)\,dx\right)y = \mu\frac{dy}{dx} + \mu p(x)y = \mu\left(\frac{dy}{dx} + p(x)y\right).$$

Thus if we multiply our original differential equation by μ it becomes

$$\mu\left(\frac{dy}{dx} + p(x)y\right) = \mu q(x)$$

but by the above calculation we can express the left side as a derivative to get

$$\frac{d}{dx}(\mu y) = \mu q(x).$$

Integrating both sides with respect to x gives

$$\mu y = \int \mu q(x)\,dx$$

and dividing through by μ we get the solution

$$y = \frac{1}{\mu}\int \mu q(x)\,dx.$$

In summary we have shown that if we set

$$\mu = e^{\int p(x)\,dx}$$

Then the solution to our differential equation is

$$y = \frac{1}{\mu}\int \mu q(x)\,dx.$$

Note that there is not a unique solution. Since an indefinite integral appears in our solution, we have an infinite family of solutions. Thus we leave our answer in terms of an arbitrary constant of integration C and such a solution is called the **general solution** to the differential equation.

▶ **Example 2** Use the method of integrating factors to find the general solution to the differential equation

$$2\frac{dy}{dx} + 4y = 1.$$

Solution We need to find $p(x)$ and $q(x)$ so we first put the differential equation in general form by dividing both sides by 2:

$$\frac{dy}{dx} + 2y = \frac{1}{2}.$$

We now see that $p(x) = 2$ and $q(x) = 1/2$ so we set

$$\mu = e^{\int 2dx} = e^{2x}$$

Then the general solution is given by

$$\begin{aligned} y &= \frac{1}{\mu}\int \mu q(x)\,dx \\ &= \frac{1}{e^{2x}}\int e^{2x}\left(\frac{1}{2}\right)\,dx \\ &= \frac{1}{2}e^{-2x}\int e^{2x}\,dx \\ &= \frac{1}{2}e^{-2x}\left(\frac{1}{2}e^{2x} + C\right) \\ &= \frac{1}{4} + Ce^{-2x}. \end{aligned}$$

Note: Perhaps you're thinking the coefficient of e^{-2x} in the final answer should be $C/2$ instead of C. The reason for this is as follows. Since C is arbitrary, so is $C/2$. Thus we replace $C/2$ in the final answer with just C itself for simplicity and this form of the general solution still describes the same family of functions.

■ SEPARATION OF VARIABLES

Another method, called separation of variables, is applicable when the differential equation can be written in the form

$$h(y)\frac{dy}{dx} = g(x).$$

In this case we say the differential equation is **separable** and we can equivalently express the differential equation in differential form:

$$h(y)\,dy = g(x)\,dx$$

If we integrate the left side with respect to y and the right side with respect to x we obtain

$$H(y) = G(x) + C$$

where $H(y)$ is an antiderivative for $h(y)$ and $G(x)$ is an antiderivative for $g(x)$. We now make the observation that y is a solution to this new equation if and only if it is a solution to original the differential equation. (You should verify this.)

Note that the equation $H(y) = G(x) + C$ only defines a solution *implicitlly* and it may not be possible to solve for y as a function of x, but we should do so if possible.

In summary, if our differential equation has the form Another method, called separation of variables, is applicable when the differential equation can be written in the form

$$h(y)\frac{dy}{dx} = g(x)$$

then integrate both sides of

$$\int h(y)\,dy = \int g(x)\,dx$$

to get the family of solutions

$$H(y) = G(x) + C$$

and solve for y if possible.

▶ **Example 3** Solve the initial value problem

$$\frac{dy}{dt} = \frac{2t+1}{2y-2}, \qquad y(0) = -1$$

First we find a general solution to the differential equation. It is separable and so we can rewrite in differential form:

$$(2y-2)\,dy = (2t+1)\,dt$$

We integrate both sides:

$$\int (2y-2)\,dy = \int (2t+1)\,dt$$

and obtain

$$y^2 - 2y = t^2 + t + C$$

We now apply the initial condition to get

$$(-1)^2 - 2(-1) = 0^2 + 0 + C$$

and hence $C = 3$. This gives the equation

$$y^2 - 2y = t^2 + t + 3$$

and this equation implicitly defines a solution to the initial value problem.